LOGARITHMS

Logarithm: For ant two positive real numbers a, b, and a ≠ 1. If the real number x such then $a^x = b$, then x is called logarithm of b to the base a. it is denoted by $\log_a b$

$$a^x = b \Leftrightarrow x = \log_a b$$

Standard formulae of logarithms:

(i) $\log_a mn = \log_a m + \log_a n$

(ii) $\log_a \frac{m}{n} = \log_a m - \log_a n$

(iii) $\log_a m^n = m \log_a n$

(iv) $\log_a a = 1$

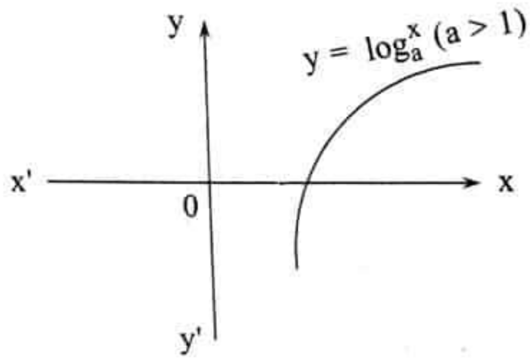
(v) $\log_a m = \log_b m \log_a b$

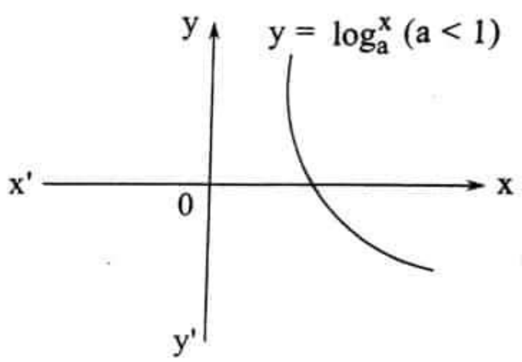
(vi) $\log_a b = \frac{1}{\log_b a}$

(vii) $a^{\log_a m} = m$

Logarithmic Function:

Let a be a positive real number and a ≠ 1. The function f: (o, ∞) → R Defined by f(x) = $\log_a x$





■ $\log_a x = \log_a y \Rightarrow x = y$

■ If $a > 1$, then $x > y \Rightarrow \log_a x > \log_a y$

■ If $0 < a < 1$, then $x > y \Rightarrow \log_a x < \log_a y$

PARTIAL FRACTIONS

Fractions:

If f(x) and g(x) are two polynomials, g(x) ≠ 0, then $\frac{f(x)}{g(x)}$ is called rational fraction.

Ex:

$\frac{4x-1}{x^2+3x+2}$, $\frac{3x^2+2}{(x-1)(3x+2)}$ etc. are rational fractions.

Proper Fraction:

A rational fraction $\frac{f(x)}{g(x)}$ is said to be a Proper fraction if the degree of g(x) is greater than the degree of f(x).

Ex:

$\frac{x+1}{x^2+3x+2}$, $\frac{3x^2+2}{x^3-1}$ etc. are the proper fractions.

Improper Fraction:

A rational fraction $\frac{f(x)}{g(x)}$ is said to be an Improper fraction if the degree of g(x) is less than the degree of f(x).

Ex:

$\frac{x^2-2x-3}{x-1}$, $\frac{x^3+2}{x+2}$ etc. are the Improper fractions.

Partial Fractions:

Expressing rational fractions as the sum of two or more simpler fractions is called resolving a given fraction into a partial fraction.

■ If R(x) = $\frac{f(x)}{g(x)}$ is proper fraction, then

Case(i): - For every factor of g(x) of the form $(ax + b)^n$, there will be a sum of n partial fractions of the form:

$$\frac{A_1}{ax+b} = \frac{A_2}{(ax+b)^2} = \frac{A_3}{(ax+b)^3} = \cdots = \frac{A_n}{(ax+b)^n}$$

Case(ii): - For every factor of g(x) of the form $(ax^2 + bx + c)^n$, there will be a sum of n partial fractions of the form:

$$\frac{A_1x+B_1}{ax^2+bx+c} = \frac{A_2x+B_2}{(ax^2+bx+c)^2} = \frac{A_3x+B_3}{(ax^2+bx+c)^3} = \cdots = \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

■ If R(x) = $\frac{f(x)}{g(x)}$ is improper fraction, then

Case (i): - If degree f(x) = degree of g(x), $\frac{f(x)}{g(x)} = k + \frac{h(x)}{g(x)}$ where k is the quotient of the highest degree term of f(x) and g(x).

Case (ii): - If f(x) > g(x)

R(x) = $\frac{f(x)}{g(x)} = Q(x) + \frac{h(x)}{g(x)}$

MATRICES AND DETERMINANTS

Matrix: A set of numbers arranged in the form of a rectangular array having rows and columns is called Matrix.

•Matrices are generally enclosed by brackets like

•Matrices are denoted by capital letters A, B, C, and so on

•Elements in a matrix are real or complex numbers; real or complex real-valued functions.

Oder of Matrix: A matrix having 'm' rows and 'n' columns is said to be of order m x n read as m by n.

Ex:

(i) $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}_{2 \times 2}$ (ii) $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 7 & 9 \end{bmatrix}_{2 \times 3}$

Types Of Matrices

Rectangular Matrix: A matrix in which the no. of rows is not equal to the no. of columns is called a rectangular matrix.

$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 5 \end{bmatrix}_{2 \times 3}$ $\begin{bmatrix} 1 & 0 \\ 2 & -4 \\ 3 & -2 \end{bmatrix}_{3 \times 2}$

Square Matrix: A matrix in which the no. of rows is equal to no. of columns is called a square matrix.

Ex: - (i) $\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}_{2 \times 2}$ (ii) $\begin{bmatrix} 0 & -3 & 3 \\ 2 & 8 & 4 \\ 1 & 7 & 1 \end{bmatrix}_{3 \times 3}$

Principal diagonal (diagonal) Matrix: If A = [a_{ij}] is a square matrix of order 'n' the elements a_{11}, a_{22}, a_{33}, ………. A_{nn} is said to constitute its principal diagonal.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 5 \\ 1 & 2 & 9 \end{bmatrix}$

Trace Matrix: The sum of the elements of the principal diagonal of a square matrix A is called the trace of the matrix. It is denoted by Tr (A).

Ex: $A = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$ $\quad$ $B = \begin{bmatrix} 1 & 5 & 5 \\ 4 & -2 & 6 \\ 1 & 0 & 3 \end{bmatrix}$

$Tr(A) = 1 + 5 = 6$ $\quad$ $Tr(B) = 1 - 2 + 3 = 2$

Diagonal Matrix: If each non-diagonal element of a square matrix is 'zero' then the matrix is called a diagonal matrix.

Ex: - $A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad$ $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Scalar Matrix: If each non-diagonal element of a square matrix is 'zero' and all diagonal elements are equal to each other, then it is called a scalar matrix.

Ex: $A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ $\quad$ $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Identity Matrix or Unit Matrix: If each of the non-diagonal elements of a square matrix is 'zero' and all diagonal elements are equal to '1', then that matrix is called unit matrix

Ex: - $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Null Matrix or Zero Matrix: If each element of a matrix is zero, then it is called a null matrix.

Ex: - $0_{2x2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad$ $0_{2x3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Row matrix & column Matrix: A matrix with only one row s called a row matrix and a matrix with only one column is called a column matrix.

Ex: - $[0 \quad 2 \quad 3] \longrightarrow$ row matrix $\quad$ $\begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \longrightarrow$ column matrix

Triangular matrices:

A square matrix $A = [a_{ij}]$ is said to be upper triangular if $a_{ij} = 0 \quad \forall\, i > j$

A square matrix $A = [a_{ij}]$ is said to be lower triangular matrix $a_{ij} = 0 \;\forall\, i < j$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \longrightarrow \text{Upper triangular matrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix} \longrightarrow \text{Lower triangular matrix}$$

Equality of matrices:

matrices A and B are said to be equal if A and B are of the same order and the corresponding elements of A and B are equal.

$$\text{Ex: -} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \Rightarrow a=p;\, b = q;\, c = r;\, d = s$$

Addition of matrices:

If A and B are two matrices of the same order, then the matrix obtained by adding the corresponding elements of A and B is called sum of A and B. It is denoted by A + B.

$$\text{Ex: } A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 5 \\ 0 & 7 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 3-2 & 0+5 \\ 2+0 & 1+7 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1 & 5 \\ 2 & 8 \end{bmatrix}$$

Subtraction Of Matrices:

If A and B are two matrices of the same order, then the matrix obtained by subtracting the corresponding elements of A and B is called the difference from A to B.

$$\text{Ex: } A = \begin{bmatrix} 3 & 7 \\ 2 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 3-2 & 7-5 \\ 2-0 & 8-3 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

Product of Matrices:

Let $A = [a_{ik}]_{mxn}$ and $B = [b_{kj}]_{nxp}$ be two matrices, then the matrix $C = [c_{ij}]_{mxp}$ where

$$C_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

Note: Matrix multiplication of two matrices is possible when no. of columns of the first matrix is equal to no. of rows of the second matrix.

$A_{m \times n} \cdot B_{p \times q} = AB_{m \times q}; n = p$

Transpose of Matrix: If $A = [a_{ij}]$ is an m x n matrix, then the matrix obtained by interchanging the rows and columns is called the transpose of A. It is denoted by A^I or A^T.

Ex: - $A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 6 & 1 \end{bmatrix}_{2\times3}$ $A^T = \begin{bmatrix} 2 & 0 \\ 3 & 6 \\ 5 & 1 \end{bmatrix}_{3\times2}$

Note: (i) $(A^I)^I = A$ (ii) $(k A^I) = k . A^I$ (iii) $(A + B)^T = A^T + B^T$ (iv) $(AB)^T = B^T A^T$

Symmetric Matrix: A square matrix A is said to be symmetric if $A^T = A$

If A is a symmetric matrix, then $A + A^T$ is symmetric.

Skew-Symmetric Matrix: A square matrix A is said to be skew-symmetric if $A^T = -A$

If A is a skew-symmetric matrix, then $A - A^T$ is skew-symmetric.

Minor of an element: Consider a square matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

the minor element in this matrix is defined as the determinant of the 2x2 matrix obtained after deleting the rows and the columns in which the element is present.

Ex: - minor of a_3 is $\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = b_1c_2 - b_2c_1$

Minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = a_1c_3 - a_3c_1$

Cofactor of an element: The cofactor of an element in i th row and j th column of A_{3x3} matrix is defined as its minor multiplied by $(-1)^{i+j}$.

Properties of determinants:

If each element of a row (column) f a square matrix is zero, then the determinant of that matrix is zero.

Ex: - $A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} \Rightarrow \det(A) = 0$

If A is a square matrix of order 3 and k is scalar then.

If two rows (columns) of a square matrix are identical (same), then Det. Of that matrix is zero.

Ex: - A = $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 5 & 6 & 8 \end{vmatrix}$ ⇒ det(A) = 0

If each element in a row (column) of a square matrix is the sum of two numbers then its determinant can be expressed as the sum of the determinants.

Ex: - $\begin{vmatrix} a+x & b & c \\ d+y & e & f \\ g+z & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} x & b & c \\ y & e & f \\ z & h & i \end{vmatrix}$

If each element of a square matrix are polynomials in x and its determinant is zero when x = a, then (x-a) is a factor of that matrix.

For any square matrix A Det(A) = Det (A^I).

Det (AB) = Det(A). Det(B).

For any positive integer n Det (A^n) = $(\text{Det}A)^n$.

Singular and non-singular matrices:

A Square matrix is said to be singular if its determinant is is zero, otherwise it is said to be non-singular matrix.

Ex: - A = $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ det(A) = 4 – 4 = 0

∴ A is singular matrix

B = $\begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix}$

Det(B) = 4 + 4 = 8≠ 0

∴ B is non-singular

Ad joint of a matrix: The transpose of the matrix formed by replacing the elements of a square matrix A with the corresponding cofactors is called the adjoint of A.

Let A = $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ and cofactor matrix of A = $\begin{bmatrix} \forall^3 & B^3 & C^3 \\ \forall^S & B^S & C^S \\ \forall^I & B^I & C^I \end{bmatrix}$

Then adj (A) = $\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

Invertible matrix: Let A be a square matrix, we say that A is invertible if there exists a matrix B such that AB =BA = I, where I is a unit matrix of the same order as A and B.

1. $(A^{-I})^{-I} = A$
2. $(A^{I})^{-I} = (A^{-I})^{I}$
3. $(AB)^{-I} = B^{-I} A^{-I}$
4. $A^{-I} = \frac{\text{adjA}}{\text{detA}}$

Trigonometry

Compound Angles

The algebraic sum of two or more angles is called a 'compound angle'. Thus, angles A + B, A – B, A + B + C etc., are Compound Angles

For any two real numbers A and B

⁕ $\sin(A + B) = \sin A \cos B + \cos A \text{ Cos } B$

⁕ $\sin(A - B) = \sin A \cos B - \cos A \text{ Cos } B$

⁕ $\cos(A + B) = \cos A \cos B - \sin A \sin B$

⁕ $\cos(A - B) = \cos A \cos B + \sin A \sin B$

⁕ $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

⁕ $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

⁕ $\cot(A + B) = \frac{\cot B \cot A - 1}{\cot B + \cot A}$

⁕ $\cot(A - B) = \frac{\cot B \cot A + 1}{\cot B - \cot A}$

⁕ $\tan(\frac{\pi}{4} + A) = \frac{1 + \tan A}{1 - \tan A}$

⁕ $\tan(\frac{\pi}{4} - A) = \frac{1 - \tan A}{1 + \tan A}$

⁕ $\cot(\frac{\pi}{4} + A) = \frac{\cot A - 1}{\cot A + 1}$

⁕ $\cot(\frac{\pi}{4} - A) = \frac{\cot A + 1}{\cot A - 1}$

⁕ $\sin(A + B + C) = \sum \sin A \cos B \cos C - \sin A \sin B \sin C$

⁕ $\cos(A+B+C) = \cos A\cos B\cos C - \sum\cos A\sin B\sin C$

⁕ $\tan(A+B+C) = \frac{\sum\tan A - \prod\tan A}{1-\sum(\tan A\tan B)}$

⁕ $\cot(A+B+C) = \frac{\sum\cot A - \prod\cot A}{1-\sum(\cot A\cot B)}$

⁕ $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

⁕ $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

	15^0	75^0	$22\,{}^1\!/_2{}^0$	$67\,{}^1\!/_2{}^0$
Sin	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$	$\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$
Cos	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$	$\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$
tan	$2-\sqrt{3}$	$2+\sqrt{3}$	$\sqrt{2}-1$	$\sqrt{2}+1$
Cot	$2+\sqrt{3}$	$2-\sqrt{3}$	$\sqrt{2}+1$	$\sqrt{2}-1$

Multiple and Sub Multiple Angles

If A is an angle, then its integral multiples 2A, 3A, 4A, … are called ‘multiple angles ‘of A and the multiple of A by fraction like $\frac{1}{2}, \frac{1}{3}$, … are called ‘submultiple angles.

⁕ $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

⁕ $\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$= \frac{1-\tan^2 A}{1+\tan^2 A}$

⁕ $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

⁕ $\cot 2A = \frac{\cot^2 A - 1}{2\cot A}$

■ If $\frac{A}{2}$ is not an add multiple of $\frac{\pi}{2}$

⁕ $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2} = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}$

※ $\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$

$= 2\cos^2 \frac{A}{2} - 1$

$= 1 - 2\sin^2 \frac{A}{2}$

$= \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}}$

※ $\tan A = \frac{2\tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$

※ $\cot A = \frac{\cot^2 \frac{A}{2} - 1}{2\cot \frac{A}{2}}$

※ $\sin A = \sqrt{\frac{1-\cos 2A}{2}}$ $\quad$ $\sin \frac{A}{2} = \sqrt{\frac{1-\cos A}{2}}$

※ $\cos A = \sqrt{\frac{1+\cos 2A}{2}}$ $\quad$ $\cos \frac{A}{2} = \sqrt{\frac{1+\cos A}{2}}$

※ $\tan A = \sqrt{\frac{1-\cos 2A}{1+\cos 2A}}$ $\quad$ $\tan \frac{A}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}}$

※ $\cot A = \sqrt{\frac{1+\cos 2A}{1-\cos 2A}}$ $\quad$ $\cot \frac{A}{2} = \sqrt{\frac{1+\cos A}{1-\cos A}}$

※ $\sin 3A = 3\sin A - 4\sin^3 A$

※ $\cos 3A = 4\cos^3 A - 3\cos A$

※ $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

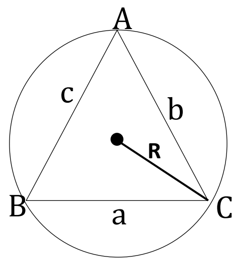
※ $\cot 3A = \frac{3\cot A - \cot^3 A}{1 - 3\cot^2 A}$

※ $\tan A + \cot A = 2\operatorname{cosec} 2A$

※ $\cot A - \tan A = 2\cot 2A$

	18	36	54	72
Sin	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$
Cos	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{4}$

PROPERTIES OF TRIANGLES



In ΔABC,
Lengths AB = c; BC = a; AC =b
Area of the tringle is denoted by Δ.
Perimeter of the triangle = 2s = a + b + c
A = ∠CAB; B = ∠ABC; C = ∠BCA.
R is circumradius.

Sine rule:

In ΔABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\Rightarrow a = 2R \sin A; \ b = 2R \sin B; \ c = 2R \sin C$$

Where R is the circum radius and a, b, c, are lengths of the sides of ΔABC.

Cosine rule:

In ΔABC,

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \Rightarrow \cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$b^2 = a^2 + c^2 - 2ac \cos B \quad \Rightarrow \cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$c^2 = a^2 + b^2 - 2ab \cos C \quad \Rightarrow \cos A = \frac{a^2+b^2-a^2}{2ab}$$

projection rule:

In ΔABC,
$a = b \cos C + c \cos B$
$b = a \cos C + c \cos A$
$c = a \cos B + b \cos A$

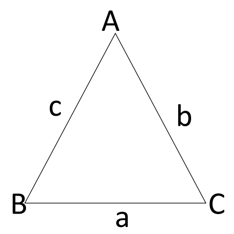
Tangent rule (Napier's analogy):

In ΔABC,

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$$

$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$$

$$\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}$$



Area of the triangle:

In ΔABC, a, b, and c are sides

$S = \frac{a+b+c}{2}$ and area of the triangle $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

HYPERBOLIC FUNCTIONS

⊗ $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \infty$

⊗ The function f: R→R defined by $f(x) = \frac{e^x - e^{-x}}{2}$ ∀ x ∈ R is called the 'hyperbolic sin' function. It is denoted by sinh x.

$\therefore \sinh x = \frac{e^x - e^{-x}}{2}$

Similarly,

⊗ $\cosh x = \frac{e^x + e^{-x}}{2}$ ∀ x ∈ R

⊗ $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ ∀ x ∈ R

⊗ $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ ∀ x ∈ R

⊗ $\text{sech}\, x = \frac{2}{e^x + e^{-x}}$ ∀ x ∈ R

⊗ $\text{cosech}\, x = \frac{2}{e^x - e^{-x}}$ ∀ x ∈ R

Identities:

⊗ $\cosh^2 x - \sinh^2 x = 1$

$\cosh^2 x = 1 + \sinh^2 x$

$\sinh^2 x = \cosh^2 x - 1$

⊗ $\text{sech}^2 x = 1 - \tanh^2 x$

$\tanh^2 x = 1 - \text{sesh}^2 x$

⊗ $\text{cosech}^2 x = \coth^2 x - 1$

$\coth^2 x = 1 + \coth^2 x$

Addition formulas of hyperbolic functions:

⊗ $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

⊗ $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

⊗ $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

⊗ $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

⊗ $\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

⊗ $\tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

⊗ $\coth(x+y) = \frac{\coth x \coth y + 1}{\coth x + \coth y}$

⊗ $\sinh 2x = 2 \sinh x \cosh 2x = \frac{2 \tanh x}{1 - \tanh^2 x}$

⊗ $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x}$

⊗ $\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$

Inverse hyperbolic functions:

⊗ $\text{Sinh}^{-1}x = \log_e\left(x + \sqrt{1 + x^2}\right)\ \forall\, x \in R$

⊗ $\text{Cosh}^{-1}x = \log_e\left(x + \sqrt{x^2 - 1}\right)\ \forall\, x \in (1, \infty)$

⊗ $\text{Tanh}^{-1}x = \frac{1}{2}\log_e\left(\left|\frac{1+x}{1-x}\right|\right)\ \forall\, |x| < 1$

Function	Domain	Range
Sinh x	R	R
Cosh x	R	$[1, \infty)$
Tanh x	R	$(-1, 1)$
Coth x	$R - \{0\}$	$(-\infty, -1) \cup (1, \infty)$
Sech x	R	$(0, 1)$
Cosech x	$R - \{0\}$	$R - \{0\}$

COMPLEX NUMBERS

The equation $x^2 + 1 = 0$ has no roots in real number system.
∴ scientists imagined a number 'i' such that $i^2 = -1$.
Complex number: if x, y are any two real numbers then the general form of the complex number is
$z = x + i\,y$; where x real part and y is imaginary part.
$3 + 4i$, $2 - 5i$, $-3 + 2i$ are the examples for Complex numbers.

* $z = x + i\,y$ can be written as (x, y).
* If $z_1 = x_1 + i\,y_1$, $z_2 = x_2 + i\,y_2$, then
  (i) $z_1 + z_2 = (x_1 + x_2, y_1 + y_2) = (x_1 + x_2) + i\,(y_1 + y_2)$
  (ii) $z_1 - z_2 = (x_1 - x_2, y_1 - y_2) = (x_1 - x_2) + i\,(y_1 - y_2)$
  (iii) $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) = (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1)$
  (iv) $z_1 / z_2 = (x_1 x_2 + y_1 y_2 / x_2^2 + y_2^2, x_2 y_1 - x_1 y_2 / x_2^2 + y_2^2)$
  $= (x_1 x_2 + y_1 y_2 / x_2^2 + y_2^2) + i\,(x_2 y_1 - x_1 y_2 / x_2^2 + y_2^2)$

Multiplicative inverse of complex number:
Multiplicative inverse of complex number z is $1/z$.
$z = x + i\,y$ then $1/z = x - i\,y / x^2 + y^2$

Conjugate complex number:

* The complex numbers $x + i\,y$, $x - i\,y$ are called conjugate complex numbers.

Conjugate of z is denoted by $\overline{z}$

* The sum and product of two conjugate complex numbers are real.
* If z_1, z_2 are two complex numbers then
  (i) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$
  (ii) $\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$
  (iii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
  (iv) $z + \overline{z} = 2\,\mathrm{Re}(z)$ and $z - \overline{z} = 2\,\mathrm{Im}(z)$

Modulus and amplitude of complex number:

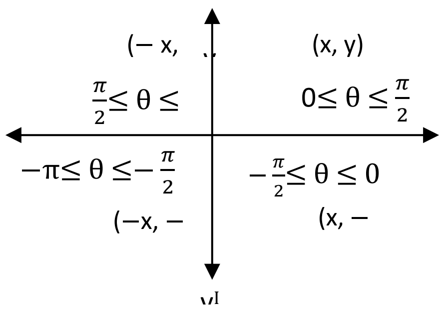
* **Modulus: -** If $z = x + i\,y$, then the non-negative real number $\sqrt{x^2 + y^2}$ is called modulus of z and it is denoted by $|z|$ or 'r'.
* **Amplitude: -** The complex number $z = x + i\,y$ represented by the point P (x, y) on the XOY plane. $\angle XOP = \theta$ is called amplitude of z or argument of z.
* $x = r\cos\theta$, $y = r\sin\theta$
  $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2(1)$
  $\Rightarrow x^2 + y^2 = r^2$
  $\Rightarrow r = \sqrt{x^2 + y^2}$ and $|z| = r$.
* $\mathrm{Arg}\,(z) = \tan^{-1}(y/x)$
* $\mathrm{Arg}\,(z_1.z_2) = \mathrm{Arg}\,(z_1) + \mathrm{Arg}\,(z_2) + n\pi$ for some $n \in \{-1, 0, 1\}$
* $\mathrm{Arg}(z_1/z_2) = \mathrm{Arg}\,(z_1) - \mathrm{Arg}\,(z_2) + n\pi$ for some $n \in \{-1, 0, 1\}$

Note:

■ $e^{i\theta} = \cos\theta + i\sin\theta$

■ $e^{-i\theta} = \cos\theta - i\sin\theta$

Argand plane: The plane contains all complex numbers is called Argand plane. This was introduced by the mathematician Gauss (1777-1855), who first thought that complex numbers can be represented as a two-dimensional plane.



Square root of complex number:

If $\sqrt{a + ib} = x + iy$, then $x = \sqrt{\frac{\sqrt{a^2+b^2}+a}{2}}$, $y = \sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}$

2.DE- MOIVERS THEOREM

De- Moiver's theorem: For any integer n and real number θ, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

→ $\cos\alpha + i\sin\alpha$ can be written as cis α

→ $\text{cis}\,\alpha.\text{cis}\,\beta = \text{cis}(\alpha + \beta)$

→ $1/\text{cis}\alpha = \text{cis}(-\alpha)$

→ $\text{cis}\alpha/\text{cis}\beta = \text{cis}(\alpha - \beta)$

⟹ $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

⟹ $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \cos^2\theta - i^2\sin^2\theta = \cos^2\theta + \sin^2\theta = 1$.

→ $\cos\theta + i\sin\theta = 1/\cos\theta - i\sin\theta$ and $\cos\theta - i\sin\theta = 1/\cos\theta + i\sin\theta$

⟹ $(\cos\theta - i\sin\theta)^n = (1/(\cos\theta -+i\sin\theta))^n = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

n^{th} root of a complex number: let n be a positive integer and $z_0 \neq 0$ be a given complex number. Any complex number z satisfying $z^n = z_0$ is called an n^{th} root of z_0. It is denoted by $z_0^{1/n}$ or $\sqrt[n]{z_0}$

⟹ let $z = r(\cos\theta + i\sin\theta) \neq 0$ and n be a positive integer. For $k \in \{0, 1, 2, 3..., (n-1)\}$ let $a_k = r^{\frac{1}{n}}\,\text{cis}\left(\frac{\theta+2k\pi}{n}\right)$. Then $a_0, a_1, a_2, ..., a_{n-1}$ are all n distinct n^{th} roots of z and any n^{th} root of z is coincided with one of them.

n^{th} root of unity: Let n be a positive integer greater than 1 and

$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, then $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are all n^{th} root of unity

Note:

- **(i)** The sum of the n^{th} roots of unity is zero.
- **(ii)** The product of n^{th} roots of unity is $(-1)^{n-1}$.
- **(iii)** The n^{th} roots of unity $1, \omega, \omega^2, ..., \omega^{n-1}$ are in geometric progression with common ratio ω.

Cube root of unity:

$x^3 - 1 = 0 \Rightarrow x^3 = 1$

$x = 1^{1/3}$

cube roots of unity are: $1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$

$$\omega = \frac{-1+\sqrt{3}i}{2}, \omega^2 = \frac{-1-\sqrt{3}i}{2}$$

$$\omega^2 + \omega + 1 = 0 \text{ and } \omega^3 = 1$$

SUM AND PRODUCT TRANSFORMATIONS

For A, B∈ R

- ⁎ $\sin(A+B)+\sin(A-B)=2\sin A\cos B$
- ⁎ $\sin(A+B)-\sin(A-B)=2\cos A\sin B$
- ⁎ $\cos(A+B)+\cos(A-B)=2\cos A\cos B$
- ⁎ $\cos(A+B)-\cos(A-B)=-2\sin A\sin B$

For any two real numbers C and D

- ⁎ $\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- ⁎ $\sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
- ⁎ $\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- ⁎ $\cos C-\cos D=-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

If $A+B+C=\pi$ or 180^0, then

- ⁎ $\sin(A+B)=\sin C$; $\sin(B+C)=\sin A$; $\sin(A+C)=\sin B$
- ⁎ $\cos(A+B)=-\cos C$; $\cos(B+C)=-\cos A$; $\cos(A+C)=-\cos B$

If $A+B+C=90^0$ or $\frac{\pi}{2}$ then

- ⁎ $\sin\left(\frac{A+B}{2}\right)=\cos\frac{C}{2}$; $\sin\left(\frac{B+C}{2}\right)=\cos\frac{A}{2}$; $\sin\left(\frac{A+C}{2}\right)=\cos\frac{B}{2}$
- ⁎ $\cos\left(\frac{A+B}{2}\right)=\sin\frac{C}{2}$; $\cos\left(\frac{B+C}{2}\right)=\sin\frac{A}{2}$; $\cos\left(\frac{A+C}{2}\right)=\sin\frac{B}{2}$

If $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$ then

- ⁎ $\sin(A+B)=\cos C$; $\sin(B+C)=\cos A$; $\sin(A+C)=\cos B$
- ⁎ $\cos(A+B)=\sin C$; $\cos(B+C)=\sin A$; $\cos(A+C)=\sin B$

INVERSE TRIGONOMETRIC FUNCTIONS

If A, B are two sets and f: A→ B is a bijection, then f^{-1} is existing and $f^{-1}: B \to A$ is an inverse function.

⊗ The function $\text{Sin}^{-1}: [-1, 1] \to [\frac{-\pi}{2}, \frac{\pi}{2}]$ is defined by $\text{Sin}^{-1} x = \theta \Leftrightarrow \theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$ and $\sin\theta = x$

⊗ The function $\text{Cos}^{-1}: [-1, 1] \to [0, \pi]$ is defined by $\text{Sin}^{-1} x = \theta \Leftrightarrow \theta \in [0, \pi]$ and $\cos\theta = x$

⊗ The function $\text{Tan}^{-1}: R \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is defined by $\text{Tan}^{-1} x = \theta \Leftrightarrow \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\theta = x$

⊗ The function $\text{Sec}^{-1}: [-\infty, -1] \cup [1, \infty] \to [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ is defined by $\text{Sin}^{-1} x = \theta \Leftrightarrow \theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ and $\sec\theta = x$

⊗ The function $\text{Cosec}^{-1}: [-\infty, -1] \cup [1, \infty] \to [\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ is defined by $\text{cosec}^{-1} x = \theta \Leftrightarrow \theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ and $\text{Cosec}\,\theta = x$

⊗ The function $\text{Cot}^{-1}: R \to (0, \pi)$ is defined by $\text{Cot}^{-1} x = \theta \Leftrightarrow \theta \in (0, \pi)$ and $\cot\theta = x$

Function	Domain	Range
Sin^{-1} x	$[-1, 1]$	$[\frac{-\pi}{2}, \frac{\pi}{2}]$
Cos^{-1} x	$[-1, 1]$	$[0, \pi]$
Tan^{-1} x	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
Cot^{-1} x	R	$(0, \pi)$
Sec^{-1} x	$(-\infty, -1] \cup [1, \infty)$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
$Cosec^{-1}$ x	$(-\infty, -1] \cup [1, \infty)$	$[\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

Properties of Inverse Trigonometric functions:

⊗ $Sin^{-1} x = Cosec^{-1}(1/x)\ \forall\ x \in [-1, 1] - \{0\}$

⊗ $Cos^{-1}x = Sec^{-1}(1/x)\ \forall\ x \in [-1, 1] - \{0\}$

⊗ $Tan^{-1} x = Cot^{-1}(1/x)$, if $x > 0$

⊗ $Tan^{-1} x = Cot^{-1}(1/x) - \pi$, if $x < 0$

⊗ $Sin^{-1}(-x) = -Sin^{-1}(x)\ \forall\ x \in [-1, 1]$

⊗ $Cos^{-1}(-x) = \pi - Cos^{-1}(x)\ \forall\ x \in [-1, 1]$

⊗ $Tan^{-1}(-x) = -Tan^{-1}(x)\ \forall\ x \in R$

⊗ $Cosec^{-1}(-x) = -Cose^{-1}(x)\ \forall\ x \in (-\infty, -1] \cup [1, \infty)$

⊗ $Sec^{-1}(-x) = \pi - Sec^{-1}(x)\ \forall\ x \in (-\infty, -1] \cup [1, \infty)$

⊗ $Cot^{-1}(-x) = \pi - Cot^{-1}(x)\ \forall\ x \in R$

(i) If $\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, then $Sin^{-1}(\sin\theta) = \theta$ and if $x \in [-1, 1]$, then $\sin(Sin^{-1}x) = x$

(ii) If $\theta \in [0, \pi]$, then $Cos^{-1}(\cos\theta) = \theta$ and if $x \in [-1, 1]$, then $\cos(Cos^{-1}x) = x$

(iii) If $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then $tan^{-1}(tann\ \theta) = \theta$ and if $x \in R$, then $\tan(Tan^{-1}x) = x$

(iv) If $\theta \in (0, \pi)$, then $Cot^{-1}(\cot\theta) = \theta$ and if $x \in R$, then $\cot(Cot^{-1}x) = x$

(v) If $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, then $Sec^{-1}(\sec\theta) = \theta$ and

if $x \in (-\infty, -1] \cup [1, \infty)$, then $\sec(Sec^{-1}x) = x$

(vi) If $\theta \in [\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$, then $Cosec^{-1}(cosec\ \theta) = \theta$ and

if $x \in (-\infty, -1] \cup [1, \infty)$, then $cosec(Cosec^{-1}x) = x$

(i) If $\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, then $Cos^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$

(ii) If $\theta \in [0, \pi]$, then $\text{Sin}^{-1}(\cos\theta) = \frac{\pi}{2} - \theta$

(iii) If $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then $\text{Cot}^{-1}(\tan\theta) = \frac{\pi}{2} - \theta$

(iv) If $\theta \in (0, \pi)$, then $\text{Tan}^{-1}(\cot\theta) = \frac{\pi}{2} - \theta$

(v) If $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, then $\text{Cosec}^{-1}(\sec\theta) = \frac{\pi}{2} - \theta$

(vi) If $\theta \in [\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$, then $\text{Sec}^{-1}(\text{cosec}\,\theta) = \frac{\pi}{2} - \theta$

⊗ $\text{Sin}^{-1}x = \text{Cos}^{-1}\sqrt{1 - x^2}$ if $0 \le x \le 1$ and $\text{Sin}^{-1}x = -\text{Cos}^{-1}\sqrt{1 - x^2}$ if $-1 \le x \le 0$

⊗ $\text{Sin}^{-1}x = \text{Tan}^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ if $x \in (-1, 1)$

⊗ $\text{Cos}^{-1}x = \text{Sin}^{-1}\sqrt{1 - x^2}$ if $x \in [0, 1]$ and $\text{Cos}^{-1}x = \pi - \text{Sin}^{-1}\sqrt{1 - x^2}$ if $x \in [-1, 0]$

⊗ $\text{Tan}^{-1}x = \text{Sin}^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \text{Cos}^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$ for $x > 0$

⊗ $\text{Cos}^{-1}x + \text{Sin}^{-1}x = \frac{\pi}{2}\ \forall\, x \in [-1, 1]$

⊗ $\text{Tan}^{-1}x + \text{Cot}^{-1}x = \frac{\pi}{2}\ \forall\, x \in R$

⊗ $\text{Sec}^{-1}x + \text{Cosec}^{-1}x = \frac{\pi}{2}\ \forall\, x \in (-\infty, -1] \cup [1, \infty)$

⊗ $\text{Sin}^{-1}x + \text{Sin}^{-1}y = \text{Sin}^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$ if $0 \le x \le 1$, $0 \le y \le 1$ and $x^2 + y^2 \le 1$

$= \pi - \text{Sin}^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$ if $0 \le x \le 1$, $0 \le y \le 1$ and $x^2 + y^2 > 1$

⊗ $\text{Cos}^{-1}x + \text{Cos}^{-1}y = \text{Cos}^{-1}(xy - \sqrt{1-y^2}\,\sqrt{1-x^2})$ if $0 \le x, y \le 1$ and $x^2 + y^2 \ge 1$

$= \pi - \text{Cos}^{-1}(xy - \sqrt{1-y^2}\,\sqrt{1-x^2})$ if $0 \le x \le 1$, $0 \le y \le 1$ and $x^2 + y^2 < 1$

⊗ $\text{Tan}^{-1}x + \text{Tan}^{-1}y = \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$ if $x > 0$, $y > 0$ and $xy < 1$

$= \pi + \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$ if $x > 0$, $y > 0$ and $xy > 1$

$= \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$ if $x < 0$, $y < 0$ and $xy > 1$

$= -\pi + \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$ if $x < 0$, $y < 0$ and $xy < 1$

⊗ $\text{Tan}^{-1}x - \text{Tan}^{-1}y = \text{Tan}^{-1}\left(\frac{x-y}{1+xy}\right)$ if $x > 0$, $y > 0$ or $x < 0$, $y < 0$

⊗ $2\,\text{Sin}^{-1}x = \text{Sin}^{-1}(2x\sqrt{1-x^2})$ if $x \le \frac{1}{\sqrt{2}}$

$= \pi - \text{Sin}^{-1}(2x\sqrt{1-x^2})$ if $x > \frac{1}{\sqrt{2}}$

⊗ $2\,\text{Cos}^{-1}x = \text{Cos}^{-1}(2x^2 - 1)$ if $x \ge \frac{1}{\sqrt{2}}$

$=Cos^{-1}(1-2x^2)$ if $x < \frac{1}{\sqrt{2}}$

⊗ $2\ Tan^{-1} x = Tan^{-1}\left(\frac{2x}{1-x^2}\right)$ if $|x| < 1$

$= \pi + Tan^{-1}\left(\frac{2x}{1-x^2}\right)$ if $|x| \geq 1$

$= Sin^{-1}\left(\frac{2x}{1+x^2}\right)$ if $x \geq 0$

$= Cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ if $x \geq 0$

⊗ $3Sin^{-1}x = Sin^{-1}(3x - 4x^3)$

⊗ $3Cos^{-1}x = Cos^{-1}(4x^3 - 3x)$

⊗ $3Tan^{-1}x = tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$

Solutions of Simultaneous Equations

Matrix Inversion Method:

Let system of simultaneous equations be

$a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$

$a_3 x + b_3 y + c_3 z = d_3$

The matrix form of the above equations is

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Co efficient Matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Variable Matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Constant Matrix $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Therefore, the matrix equation is $AX = B$

If $DetA \neq 0$, A^{-1} is exists

$X = A^{-1} B$

By using above Condition, we get the values of x, y and z

This Method is called as Matrix Inversion Method

Cramer's Method:

Let system of simultaneous equations be

$a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$

$a_3 x + b_3 y + c_3 z = d_3$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

Δ_1 is obtained by replacing the coefficients of x (1^{st} column elements of Δ) by constant values

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

Δ_2 is obtained by replacing the coefficients of y (2^{nd} column elements of Δ) by constant values

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Δ_3 is obtained by replacing the coefficients of z (3^{rd} column elements of Δ) by constant values

Now $x = \frac{\Delta_1}{\Delta}$; $y = \frac{\Delta_2}{\Delta}$; $z = \frac{\Delta_3}{\Delta}$

This method is called Cramer's Method

Gauss-Jordan Method:

Let system of simultaneous equations be

$a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$

$a_3 x + b_3 y + c_3 z = d_3$

Augmented matrix: The coefficient matrix (A) augmented with the constant column matrix (B) is called augmented matrix. It is denoted by [AD].

$$[AD] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

This Matrix is reduced to the standard form of $\begin{bmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & \beta \\ 0 & 0 & 1 & \gamma \end{bmatrix}$ by using row operations

1. Interchanging any two rows
2. Multiplying the elements of any two elements of any two elements by a constant.
3. Adding to the elements of one row with the corresponding elements of another row multiplied by a constant.

∴ The solution of given system of simultaneous equations is $x = \alpha$, $y = \beta$ and $z = \gamma$.

Procedure to get standard form:

1. Take the coefficient of x as the unity as first equation.
2. If 1 is there in first row first column, then make remaining two elements in first column as zero.
3. After that if one element in R_2 or R_3 is 1, then make the remaining two elements in that column C_2 or C_3 as zeroes.
4. If any row contains two elements as zeros and only non-zero divide that row elements with the non-zero element to get unity and make the remaining two elements in that column as zeros.