MAXIMA AND MINIMA

⊗ Let f be a real function on an interval I then f is said to be

(i) an increasing function on I if
$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \ \forall \ x_1, x_2 \in I$

(ii) decreasing function on I if
$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \ \forall \ x_1, x_2 \in I$

⊗ Let f be a real function on an interval I then f is said to be

(i) strictly increasing function on I if
$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \ \forall \ x_1, x_2 \in I$

(ii) strictly decreasing function on I if
$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \ \forall \ x_1, x_2 \in I$

⊗ Let f(x) be a real valued function defined on I = (a, b) or [a, b) or (a, b] or [a, b]. Suppose f is continuous on I and differentiable in (a, b). If

(i) $f'(c) > 0 \ \forall \ c \in (a, b)$, then f is strictly increasing on I
(ii) $f'(c) < 0 \ \forall \ c \in (a, b)$, then f is strictly decreasing on I
(iii) $f'(c) \geq 0 \ \forall \ c \in (a, b)$, then f is increasing on I
(iv) $f'(c) \leq 0 \ \forall \ c \in (a, b)$, then f is decreasing on I

Critical point:

A point x = c in the domain of the function said to be 'critical point' of the function f if either $f'(c) = 0$ or $f'(c)$ does not exists.

Stationary point:

A point x = c in the domain of the function said to be 'stationary point' of the function f if $f'(c) = 0$.

MAXIMA & MINIMA

Global maxima – Global minima:

Let D be an interval in R and f: D → R be a real function and c ∈ D. Then f is said to be

(i) a global maximum on D if $f(c) \geq f(x)$
(ii) a global minimum on D if $f(c) \leq f(x)$

Relative maximum:

Let D be an interval in R and f: D → R be a real function and c ∈ D. Then f is said to be relative maximum at c if there exist $\delta > 0$ such that $f(c) \geq f(x) \ \forall \ x \in (c - \delta, c + \delta)$.

Here, f (c) is called relative maximum value of f(x) at $x = c$ and the point $x = c$ is called point of relative maximum.

Relative minimum:

Let D be an interval in R and $f: D \to R$ be a real function and $c \in D$. Then f is said to be relative maximum at c if there exist $\delta > 0$ such that $f(c) \leq f(x)\ \forall\ x \in (c - \delta, c + \delta)$.

Here, f (c) is called relative maximum value of f(x) at $x = c$ and the point $x = c$ is called point of relative minimum.

⊗ The relative maximum and minimum value of f are called extreme values.

⊗ If f is either minima or maxima $f'(\alpha) = 0$.

⊗ Let f be a continuous function om [a, b] and $\alpha \in (a, b)$

(i) if $f'(\alpha) = 0$ and $f''(\alpha) > 0$, then $f(\alpha)$ is relative minimum.

if $f'(\alpha) = 0$ and $f''(\alpha) < 0$, then $f(\alpha)$ is relative maximum

Solved Problems :

1. Find the interval in which the function $f(x) = x^2 + 2x - 5$ is increasing

Sol: Given function is $f(x) = x^2 + 2x - 5$

$f'(x) = \frac{d}{dx}(x^2 + 2x - 5) = 2x + 2$

If f(x) is increasing, then $f'(x) > 0$

$\Rightarrow 2x + 2 > 0$

$\Rightarrow 2x > -2$

$\Rightarrow x > -1$

$\therefore$ f(x) is increasing when $x > -1$

2. Find the interval in which the function $f(x) = x\, e^x$ is decreasing

Sol: Given function is $f(x) = x\, e^x$

$f'(x) = \frac{d}{dx}(x\, e^x) = x\, e^x + e^x = e^x\,(x + 1)$

If f(x) is decreasing, then $f'(x) < 0$

$\Rightarrow e^x\,(x + 1) < 0$

$\Rightarrow (x + 1) < 0$

$\Rightarrow x < -1$

$\therefore$ f(x) is decreasing when $x < -1$

3. Find the value of x of the function which $y = x^3 - 4x^2 + 5x$ is maximum

Sol: Given function is $y = x^3 - 4x^2 + 5x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 4x^2 + 5x) = 3x^2 - 4(2x) + 5 = 3x^2 - 8x + 5$$

For maxima or minima $\frac{dy}{dx} = 0$

$\Longrightarrow 3x^2 - 8x + 5 = 0$

$3x^2 - 3x - 5x + 5 = 0$

$3x(x - 1) - 5(x - 1) = 0$

$(x - 1)(3x - 5) = 0$

$x = 1$ or $x = \frac{5}{3}$

now

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2 - 8x + 5) = 6x - 8$$

When $x = 1$, $\frac{d^2y}{dx^2} = 6(1) - 8 = 6 - 8 = -2 < 0$

$\therefore$ Given function is maximum when $x = 1$

4. Find the maximum and minimum values of the function $2x^3 - 6x^2 - 18x + 21$

Sol: let $y = 2x^3 - 6x^2 - 18x + 21$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3 - 6x^2 - 18x + 21) = 6x^2 - 6(2x) - 18 = 6x^2 - 12x - 18$$

For maxima or minima $\frac{dy}{dx} = 0$

$\Longrightarrow 6x^2 - 12x - 18 = 0$

$6(x^2 - 2x - 3) = 0$

$x^2 - 2x - 3 = 0$

$x^2 - 3x + x - 3 = 0$

$x(x - 3) + 1(x - 3) = 0$

$(x - 3)(x + 1) = 0$

$x = -1$ or $x = 3$

now

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(6x^2 - 12x - 18) = 12x - 12$$

When $x = -1$, $\frac{d^2y}{dx^2} = 12(-1) - 12 = -12 - 12 = -24 < 0$

$\therefore$ Given function is maximum when $x = -1$

When $x =$, $\frac{d^2y}{dx^2} = 12(3) - 12 = 36 - 12 = 24 > 0$

$\therefore$ Given function is minimum when $x = 3$

5. Find the maximum and minimum values of the function $2x^3 - 6x^2 - 18x + 21$

Sol: let $y = 2x^3 - 6x^2 - 18x + 21$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3 - 6x^2 - 18x + 21) = 6x^2 - 6(2x) - 18 = 6x^2 - 12x - 18$$

For maxima or minima $\frac{dy}{dx} = 0$

$\Longrightarrow 6x^2 - 12x - 18 = 0$

$6(x^2 - 2x - 3) = 0$

$x^2 - 2x - 3 = 0$

$x^2 - 3x + x - 3 = 0$

$x(x - 3) + 1(x - 3) = 0$

$(x - 3)(x + 1) = 0$

$x = -1$ or $x = 3$

now

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(6x^2 - 12x - 18) = 12x - 12$$

When $x = -1$, $\frac{d^2y}{dx^2} = 12(-1) - 12 = -12 - 12 = -24 < 0$

$\therefore$ Given function is maximum when $x = -1$

When $x = 3$, $\frac{d^2y}{dx^2} = 12(3) - 12 = 36 - 12 = 24 > 0$

$\therefore$ Given function is minimum when $x = 3$

6. Find the maximum and minimum values of the function $x^3 - 6x^2 + 9x + 1$

Sol: let $y = x^3 - 6x^2 + 9x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 6x^2 + 9x + 1) = 3x^2 - 6(2x) + 9 = 3x^2 - 12x + 9$$

For maxima or minima $\frac{dy}{dx} = 0$

$\Longrightarrow 3x^2 - 12x + 9 = 0$

$3(x^2 - 4x + 3) = 0$

$x^2 - 4x + 3 = 0$

$x^2 - 3x - x - 3 = 0$

$x(x - 3) - 1(x - 3) = 0$

$(x - 3)(x - 1) = 0$

$x = 1$ or $x = 3$

now

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2 - 12x + 9) = 6x - 12$$

When $x = 1$, $\frac{d^2y}{dx^2} = 6(1) - 12 = 6 - 12 = -6 < 0$

$\therefore$ Given function is maximum when $x = 1$

Maximum value $= (1)^3 - 6(1)^2 + 9(1) + 1 = 1 - 6 + 9 + 1 = 11 - 6 = 5$

$\therefore$ The maximum value is 5

When $x =$, $\frac{d^2y}{dx^2} = 6(3) - 12 = 18 - 12 = 6 > 0$

$\therefore$ Given function is minimum when $x = 3$

Minimum value $= (3)^3 - 6(3)^2 + 9(3) + 1 = 27 - 54 + 27 + 1 = 1$

$\therefore$ The minimum value is 1

7. Find the maximum and minimum values of the function $2x^3—9x^2 +12x + 15$

Sol: let $y = 2x^3 - 9x^2 + 12x + 15$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3 - 9x^2 + 12x + 15) = 2(3x^2) - 9(2x) + 12 = 6x^2 - 18x + 12$$

For maxima or minima $\frac{dy}{dx} = 0$

$\Rightarrow 6x^2 - 18x + 12 = 0$

$6(x^2 - 3x + 2) = 0$

$x^2 - 3x + 2 = 0$

$x^2 - 2x - x - 3 = 0$

$x(x - 2) - 1(x - 2) = 0$

$(x - 2)(x - 1) = 0$

$x = 1$ or $x = 2$

now

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(6x^2 - 18x + 12) = 12x - 18$$

When $x = 1, \frac{d^2y}{dx^2} = 12(1) - 18 = 12 - 18 = -6 < 0$

$\therefore$ Given function is maximum when $x = 1$

Maximum value $= 2(1)^3 - 9(1)^2 + 12(1) + 15 = 2 - 9 + 12 + 15 = 29 - 9 = 20$

$\therefore$ The maximum value is 20

When $x = 2, \frac{d^2y}{dx^2} = 12(2) - 18 = 24 - 18 = 6 > 0$

$\therefore$ Given function is minimum when $x = 2$

Minimum value $= 2(2)^3 - 9(2)^2 + 12(2) + 15$

$= 2(8) - 9(4) + 24 + 15$

$= 16 - 36 + 39 = 19$

$\therefore$ The minimum value is 19

8. Find the maximum and minimum values of the function $4x^3 + 9x^2 - 12x + 1$

Sol: let $y = 4x^3 + 9x^2 - 12x + 1$

$$\frac{dy}{dx} = \frac{d}{dx}(4x^3 + 9x^2 - 12x + 1) = 4(3x^2) + 9(2x) - 12 = 12x^2 + 18x - 12$$

For maxima or minima $\frac{dy}{dx} = 0$

$\Rightarrow 12x^2 + 18x - 12 = 0$

$6(2x^2 + 3x - 2) = 0$

$2x^2 + 3x - 2 = 0$

$2x^2 + 4x - x - 2 = 0$

$2x(x + 2) - 1(x + 2) = 0$

$(x + 2)(2x - 1) = 0$

$x = -2$ or $x = \frac{1}{2}$

now

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(12x^2 + 18x - 12) = 24x + 18$$

When $x = -2, \frac{d^2y}{dx^2} = 24(-2) + 18 = -48 + 18 = -30 < 0$

∴ Given function is maximum when $x = -2$

Maximum value $= 4(-2)^3 + 9(-2)^2 - 12(-2) + 1$

$= 4(-8) + 9(4) + 24 + 1 = -32 + 36 + 25 = 29$

∴ The maximum value is 29

When $x = \frac{1}{2}, \frac{d^2y}{dx^2} = 24(\frac{1}{2}) + 18 = 12 + 18 = 30 > 0$

∴ Given function is minimum when $x = \frac{1}{2}$

Minimum value $= 4(\frac{1}{2})^3 + 9(\frac{1}{2})^2 - 12(\frac{1}{2}) + 1 = 4 \times \frac{1}{8} + \frac{9}{4} - 6 + 1 \quad = \frac{1}{2} + \frac{9}{4} - 6 + 1$

$= \frac{12+9-24+4}{4} = \frac{25-24}{4} = \frac{1}{4}$

∴ The minimum value is $\frac{1}{4}$

9. The sum of two numbers is 10. Find the numbers so that the sum of the squares is minimum.

Sol: Let two numbers be x and y

$\Rightarrow x + y = 10 \Rightarrow y = 10 - x$

Let S be the sum of the square of two numbers

$S = x^2 + y^2$

$= x^2 + (10 - x)^2$

$= x^2 + 100 + x^2 - 20x$

$= 2x^2 - 20x + 100$

$\frac{dS}{dx} = \frac{d}{dx}(2x^2 - 20x + 100) = 2(2x) - 20 = 4x - 20$

If S is minimum, then $\frac{dS}{dx} = 0$

$\Rightarrow 4x - 20 = 0$

$4x = 20$

$x = 5$

now

$\frac{d^2S}{dx^2} = \frac{d}{dx}(4x - 20) = 4 > 0$

∴ S is minimum when $x = 5$

$y = 10 - x = 10 - 5 = 5$

∴ The numbers are 5, 5

10. The sum of two numbers is 72. Find the numbers so that so that the sum of the squares is minimum.

Sol: Let two numbers be x and y

$\Rightarrow x + y = 72 \Rightarrow y = 72 - x$

Let S be the sum of the square of two numbers

$S = x^2 + y^2$

$= x^2 + (72 - x)^2$

$= x^2 + 5184 + x^2 - 144x$

$= 2x^2 - 144x + 5184$

$\frac{dS}{dx} = \frac{d}{dx}(2x^2 - 144x + 5184) = 2(2x) - 144 = 4x - 144$

If S is minimum, then $\frac{dS}{dx} = 0$

$\Rightarrow 4x - 144 = 0$

$4x = 144$

$x = 36$

now

$\frac{d^2S}{dx^2} = \frac{d}{dx}(4x - 144) = 4 > 0$

$\therefore$ S is minimum when $x = 36$

$y = 72 - x = 72 - 36 = 36$

$\therefore$ The numbers are 36, 36

11. The sum of two numbers is 24. Find the numbers so that so that the sum of the squares is minimum.

Sol: Let two numbers be x and y

$\Rightarrow x + y = 24 \Rightarrow y = 24 - x$

Let S be the sum of the square of two numbers

$S = x^2 + y^2$

$= x^2 + (24 - x)^2$

$= x^2 + 576 + x^2 - 48x$

$= 2x^2 - 48x + 576$

$\frac{dS}{dx} = \frac{d}{dx}(2x^2 - 48x + 576) = 2(2x) - 48 = 4x - 48$

If S is minimum, then $\frac{dS}{dx} = 0$

$\Rightarrow 4x - 48 = 0$

$4x = 48$

$x = 12$

now

$\frac{d^2S}{dx^2} = \frac{d}{dx}(4x - 48) = 4 > 0$

$\therefore$ S is minimum when $x = 12$

$y = 24 - x = 24 - 12 = 12$

$\therefore$ The numbers are 12, 12

12. The sum of two numbers is 32. Find the numbers so that their product is maximum.

Sol: Let two numbers be x and y

$\Rightarrow x + y = 32 \Rightarrow y = 32 - x$

Let P be the product of two numbers

$P = x\,y$

$= x(32 - x) = 32x - x^2$

$\frac{dS}{dx} = \frac{d}{dx}(32x - x^2) = 32 - 2x$

If P is maximum, then $\frac{dS}{dx} = 0$

$\Rightarrow 32 - 2x = 0$

$2x = 32$

$x = 16$

now

$\frac{d^2S}{dx^2} = \frac{d}{dx}(32 - 2x) = -2 < 0$

$\therefore$ P is maximum when $x = 16$

$y = 32 - x = 32 - 16 = 16$

$\therefore$ The numbers are 16, 16

13. The sum of two numbers is 26. Find them the product is to be maximum.

Sol: Let two numbers be x and y

$\Rightarrow x + y = 26 \Rightarrow y = 26 - x$

Let P be the product of two numbers

$P = x\,y$

$= x(26 - x) = 26x - x^2$

$\frac{dS}{dx} = \frac{d}{dx}(26x - x^2) = 26 - 2x$

If P is maximum, then $\frac{dS}{dx} = 0$

$\Rightarrow 26 - 2x = 0$

$2x = 26$

$x = 13$

now

$\frac{d^2S}{dx^2} = \frac{d}{dx}(26 - 2x) = -2 < 0$

$\therefore$ P is maximum when $x = 13$

$y = 26 - x = 26 - 13 = 13$

$\therefore$ The numbers are 13, 13

14. The sum of two numbers is 24. Find them the product is to be maximum.

Sol: Let two numbers be x and y

$\Rightarrow x + y = 24 \Rightarrow y = 24 - x$

Let P be the product of two numbers

$P = x\,y$

$= x(24 - x) = 24x - x^2$

$\frac{dS}{dx} = \frac{d}{dx}(24x - x^2) = 24 - 2x$

If P is maximum, then $\frac{dS}{dx} = 0$

$\Rightarrow 24 - 2x = 0$

$2x = 24$

$x = 12$

now

$\frac{d^2S}{dx^2} = \frac{d}{dx}(24 - 2x) = -2 < 0$

∴ P is maximum when $x = 12$

$y = 24 - x = 24 - 12 = 12$

∴ The numbers are 12, 12

15. Show that the maximum rectangle that can be inscribed in a circle is a square.

Sol: Let x be the length of the rectangle, y be the breadth, area of the rectangle is A and r be the radius of the circle

If the maximum rectangle is inscribed in a circle, then the diagonal of rectangle becomes the diameter of circle

Length of the diagonal $= \sqrt{x^2 + y^2}$

$\Rightarrow 2r = \sqrt{x^2 + y^2}$

$4r^2 = x^2 + y^2$

$y^2 = 4r^2 - x^2$

$y = \sqrt{4r^2 - x^2}$

area of the rectangle $= A = xy = x\sqrt{4r^2 - x^2}$

$A = x\sqrt{4r^2 - x^2}$

$\frac{dA}{dx} = \frac{d}{dx}\left(x\sqrt{x^2 - 4r^2}\right) = x\frac{1}{2\sqrt{4r^2-x^2}}\frac{d}{dx}(4r^2 - x^2) + \sqrt{4r^2 - x^2}$

$= \frac{x(-2x)}{2\sqrt{4r^2-x^2}} + \sqrt{4r^2 - x^2} = \frac{-x^2+4r^2-x^2}{\sqrt{4r^2-x^2}} = \frac{4r^2 - 2x^2}{\sqrt{4r^2-x^2}}$

If A is maximum $\frac{dA}{dx} = 0$

$\frac{4r^2 - 2x^2}{\sqrt{4r^2-x^2}} = 0$

$4r^2 - 2x^2 = 0$

$x^2 = 2r^2$

$x = \sqrt{2}\,r$

If A is maximum then, $x = \sqrt{2}\,r$ and $y = \sqrt{4r^2 - x^2} = \sqrt{4r^2 - 2r^2} = \sqrt{2r^2} = \sqrt{2}\,r$

$\Rightarrow x = \sqrt{2}\,r$ and $y = \sqrt{2}\,r$

∴ The rectangle is a square

16. Find the dimensions of rectangle of maximum area having a perimeter of 36 feet.

Sol: Let x be the length of the rectangle, y be the breadth, perimeter of the rectangle is P and area A

$P = 36$

$2x + 2y = 36$

$x + y = 18$

$y = 18 - x$

area of the rectangle $= A = xy = x(18 - x) = 18x - x^2$

$A = 18x - x^2$

$\frac{dA}{dx} = 18 - 2x$

If A is maximum $\frac{dA}{dx} = 0$

$\Longrightarrow 18 - 2x = 0$

$18 = 2x$

$x = 9$

$\frac{d^2P}{dx^2} = -2 < 0$

A is maximum when $x = 9$

$y = 18 - 9 = 9$

∴ length of rectangle = 9 ft. and breadth = 9 ft.

17. Find the dimensions of rectangle of maximum area having a perimeter of 48 feet.

Sol: Let x be the length of the rectangle, y be the breadth, perimeter of the rectangle is P and area A

$P = 48$

$2x + 2y = 48$

$x + y = 24$

$y = 24 - x$

area of the rectangle $= A = xy = x(24 - x) = 24x - x^2$

$A = 24x - x^2$

$\frac{dA}{dx} = 24 - 2x$

If A is maximum $\frac{dA}{dx} = 0$

$\Longrightarrow 24 - 2x = 0$

$24 = 2x$

$x = 12$

$\frac{d^2P}{dx^2} = -2 < 0$

A is maximum when $x = 12$

$y = 24 - 12 = 12$

∴ length of rectangle = 12 ft. and breadth = 12 ft.

18. Show that the area of rectangle of a given fixed perimeter is maximum when the rectangle is square

Sol: Let x be the length of the rectangle, y be the breadth, perimeter of the rectangle is P and area A

If perimeter of Rectangle = P = K(fixed)

$2x + 2y = K$

$x + y = \frac{K}{2}$

$y = \frac{K}{2} - x$

area of the rectangle $= A = xy = x\,(\frac{K}{2} - x) = \frac{K}{2}x - x^2$

$A = \frac{K}{2}x - x^2$

$\frac{dA}{dx} = \frac{K}{2} - 2x$

If A is maximum $\frac{dA}{dx} = 0$

$\Longrightarrow \frac{K}{2} - 2x = 0$

$\frac{K}{2} = 2x$

$x = \frac{K}{4}$

$\frac{d^2P}{dx^2} = -2 < 0$

A is maximum when $x = \frac{K}{4}$

$y = \frac{K}{2} - \frac{K}{4} = \frac{K}{4}$

$\therefore$ length of rectangle $= \frac{K}{4}$ and breadth $= \frac{K}{4}$

$\therefore$ Rectangle is a square

19. A wire of length 40 cm is bent so as to form a rectangle. If its area is to be maximum, find the dimensions of the rectangle.

Sol: Let x be the length of the rectangle, y be the breadth, perimeter of the rectangle is P and area A

$2x + 2y = 40$

$x + y = 20$

$y = 20 - x$

area of the rectangle $= A = xy = x\,(20 - x) = 20x - x^2$

$A = 20x - x^2$

$\frac{dA}{dx} = 20 - 2x$

If A is maximum $\frac{dA}{dx} = 0$

$\Longrightarrow 20 - 2x = 0$

$20 = 2x$

$x = 10$

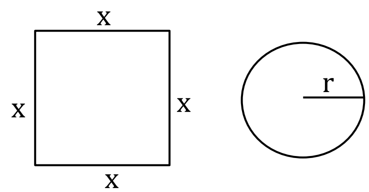
$\frac{d^2P}{dx^2} = -2 < 0$

A is maximum when $x = 10$

$y = 20 - 10 = 10$

$\therefore$ length of rectangle $= 10$ and breadth $= 10$

20. A wire of length 20 cm is to be cut into two pieces. One of the pieces is to made into a square and other into a circle. Where should the wire cut so that the combined area is minimum.



Sol: Let x be the side of the square and

r be the radius of the circle

length of the wire $= 4x + 2\pi r$

$\Longrightarrow 20 = 4x + 2\pi r$

$4x = 20 - 2\pi r$

$x = \frac{20-2\pi r}{4}$

let combined area of square and circle = A

$A = x^2 + \pi r^2$

$= \left(\frac{20-2\pi r}{4}\right)^2 + \pi r^2$

$\frac{dA}{dr} = \frac{2(20-2\pi r)(-2\pi)}{16} + 2\pi r$

$= \frac{-\pi\,(20-2\pi r)}{4} + 2\pi r$

If A is minimum $\frac{dA}{dx} = 0$

$\frac{-\pi\,(20-2\pi r)}{4} + 2\pi r = 0$

$2\pi r = \frac{\pi\,(20-2\pi r)}{4}$

$2\pi r = \frac{20\pi}{4} - \frac{2\pi^2 r}{4}$

$2\pi r + \frac{2\pi^2 r}{4} = 5\pi$

$2\pi r\left(1 + \frac{\pi}{4}\right) = 5\pi$

$2\pi r\left(\frac{\pi+4}{4}\right) = 5\pi$

$r = \frac{10}{\pi+4}$

$\frac{d^2A}{dr^2} = \frac{\pi(2\pi)}{4} + 2\pi = \frac{\pi(\pi+4)}{2} > 0$

A is minimum when $r = \frac{10}{\pi+4}$

$x = \frac{20-2\pi\frac{10}{\pi+4}}{4} = \frac{20(\pi+4)-20\pi}{4(\pi+4)} = \frac{20}{\pi+4}$

One part $= 4x = 4 \times \frac{10}{\pi+4} = \frac{80}{\pi+4}$

Another part $= 2\pi r = 2\pi \times \frac{10}{\pi+4} = \frac{20\pi}{\pi+4}$

21. A rectangular metal sheet is 8 cm long and 5 cm wide. Equal squares are cut off from the corners and the faps are then folded up to form an open box. Find the side of the square cut off so that the box may be of greater capacity. What is the maximum capacity of the box?

Sol: Given a rectangular metal sheet of length 8cm and wide 5 cm.

Length of the square x is removed from each corner

Length of box = (8 – 2x) cm

Breadth of box = (5 – 2x) cm

Height of box = x cm

Volume of box = $V = (8-2x)(5-2x)x \text{ cm}^3$

$= x(40 - 16x - 10x + 4x^2)$

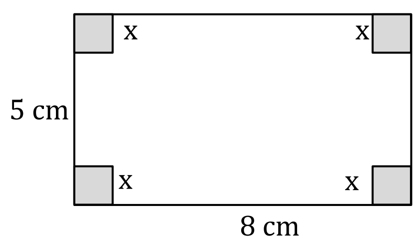
$= x(40 - 26x + 4x^2)$

$V = 4x^3 - 26x^2 + 40x$

$\frac{dV}{dx} = 4(3x^2) - 26(2x) + 40$

$= 12x^2 - 52x + 40$

For maximum volume $\frac{dV}{dx} = 0$



$\Rightarrow 12x^2 - 52x + 40 = 0$

$4(3x^2 - 13x + 10) = 0$

$3x^2 - 13x + 10 = 0$

$3x^2 - 3x - 10x + 10 = 0$

$3x(x-1) - 10(x-1) = 0$

$(x-1)(3x-10) = 0$

$x = 1$ or $x = \frac{10}{3}$

if $x = \frac{10}{3}$, then breadth of rectangle $= 5 - 2\times\frac{10}{3} = \frac{15-20}{3} = \frac{-5}{3}$

$\Rightarrow x = \frac{10}{3}$ not exist

$\therefore x = 1$

Now

$\frac{d^2V}{dx^2} = 24x - 52$

When $x = 1$, $\frac{d^2V}{dx^2} = 24\times1 - 52 = 24 - 52 = -28 < 0$

$\therefore$ Volume box is maximum at $x = 1$

Maximum volume $= 4(1)^3 - 26(1)^2 + 40(1) = 4 - 26 + 40 = 44 - 26 = 18$

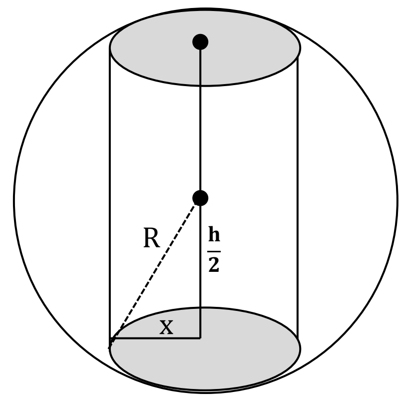
$\therefore$ side of square cut off is 1cm

$\therefore$ Maximum volume is 18 cm^3

22. A right circular cylinder is inscribed in a sphere of radius R. Show that the volume is maximum when its height is $\frac{2R}{\sqrt{3}}$

Sol: Let x be the radius of cylinder, h be the height of cylinder and the volume be V

Radius of sphere = R



From the figure $x^2 + \left(\frac{h}{2}\right)^2 = R^2$

$\Rightarrow x^2 = R^2 - \frac{h^2}{4}$

Volume of cylinder = $V = \pi x^2 h$

$$V = \pi\left(R^2 - \frac{h^2}{4}\right)h$$

$$= \pi R^2 h - \frac{\pi h^3}{4}$$

$\frac{dV}{dh} = \pi R^2 - \frac{3\pi h^2}{4}$

If V Is maximum, then $\frac{dV}{dh} = 0$

$\pi R^2 - \frac{3\pi h^2}{4} = 0$

$\pi R^2 = \frac{3\pi h^2}{4}$

$4R^2 = 3h^2$

$h^2 = \frac{4}{3}R^2$

$h = \frac{2R}{\sqrt{3}}$

$\frac{d^2V}{dh^2} = -\frac{3\pi .2h}{4} = -\frac{3\pi h}{2} < 0$

∴ Volume of cylinder is maximum when its height is $\frac{2R}{\sqrt{3}}$