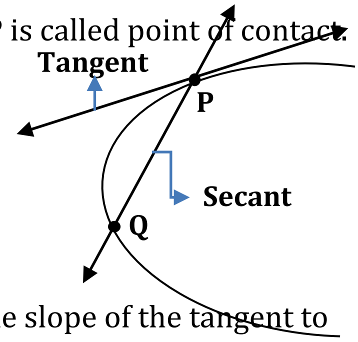
9. TANGENTS AND NORMALS

Tangent of a Curve:

Let y = f(x) be a curve, P a point on the curve. If Q(≠P) is another point on the curve then the line PQ is called secant line.

If the secant line PQ approaches to the same position as Q moves along the curve and approaches to either side then limiting position is called a 'Tangent line' to the curve at P. The point P is called point of contact.



Gradient of a curve:

Let y = f(x) be a curve and P (x, y) be a point on the curve. The slope of the tangent to the curve y = f(x) at P is called gradient of the curve.

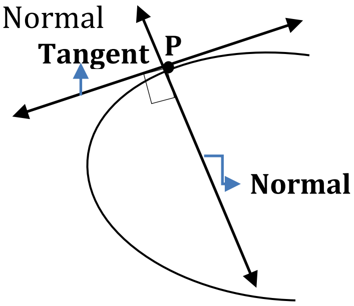
Slope of the tangent to the curve y = f(x) at P (x, y) is $m = \left(\frac{dy}{dx}\right)_{P(x,y)}$

■ The equation of the tangent at P (x_1, y_1) to the curve is

$$y - y_1 = m\,(x - x_1) \text{ where } m = \left(\frac{dy}{dx}\right)_{P(x_1,y_1)}$$

Normal of a curve:

Let y = f(x) be a curve and P (x, y) be a point on the curve. The line passing through P and perpendicular to the tangent of the curve y = f(x) at P is called Normal of the curve.

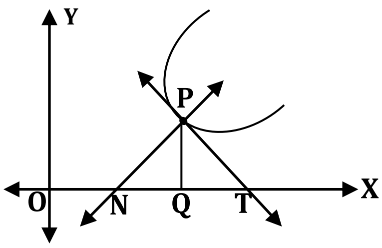


Slope of the normal is -1/m. where $m = \left(\frac{dy}{dx}\right)_{P(x_1,y_1)}$

■ The equation of the Normal at P (x_1, y_1) to the curve is

$$y - y_1 = \frac{-1}{m}(x - x_1).$$

Lengths of tangent, normal, subtangent and subnormal: -*



PT → Tangent; QT → subtangent
PN → Normal; QN → subnormal

■ if $m = \left(\frac{dy}{dx}\right)_{P(x,y)}$

(i) Length of the tangent = $\left|\frac{y\sqrt{1+m^2}}{m}\right|$

(ii) Length of the tangent = $\left|y\sqrt{1+m^2}\right|$

(iii) Length of the tangent = $\left|\frac{y}{m}\right|$

(iv) Length of the tangent = $|y\,.\,m|$

Angle between two curves:

If two curves intersect at a point P., then the angle between the tangents of the curves at P is called the angle between the curves at P.

■ If m_1, m_2 are the slopes of two tangents of the two curves and θ is the angle between the curves then

$\text{Tan}\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

Note:

(i) If $m_1 = m_2$, then two corves are touch each other.

(ii) if $m_1 \times m_2 = -1$, then two curves intersect orthogonally.

10. RATE MEASURE

Average rate of change:

if $y = f(x)$ then the average rate of change in y between $x = x_1$ and $x = x_2$ is defined as

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Instantaneous rate of change:

if $y = f(x)$, then the instantaneous rate of change of a unction f at $x = x_0$ is defined as

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Rectilinear Motion:

A motion of a particle in a line is called rectilinear motion. The rectilinear motion is denoted by $s = f(t)$ where $f(t)$ is the rule connecting 's' and 't'.

Velocity, Acceleration:

A particle starts from a fixed point and moves a distance 'S' along a straight-line during time 't' then

Velocity $= v = \frac{ds}{dt}$

Acceleration $= a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Note:

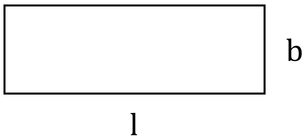
(i) if $v > 0$, then the particle s moving away from the straight point.

(ii) If $v < 0$, then particle s moving away towards the straight point.

(iii) If $v = 0$, then the particle comes rest.

The Following formulae will be used in Solving problems

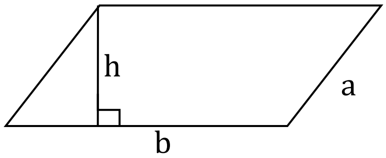
Rectangle:



Perimeter of Rectangle = 2 (l + b) units

Area of Rectangle = l × b square units

Parallelogram:



Perimeter of Parallelogram = 2 (a + b) units

Area of Parallelogram = b × h square units

Triangle:

Area of triangle $= \frac{1}{2}$ b × h square units

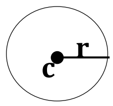
Area of Equilateral triangle $= \frac{\sqrt{3}}{4} a^2$ square units

If a, b and c are sides of a triangle

$$s = \frac{a + b + c}{2}$$

Area of Triangle $= \sqrt{s(s-a)(s-b)(s-c)}$ square units

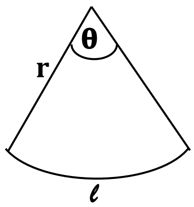
CIRCLE:



If 'r' is radius, 'd' is diameter 'P' is the perimeter or circumference and A is area of the circle then

d= 2r, P = 2πr = πd and A = πr²sq.u

SECTOR:



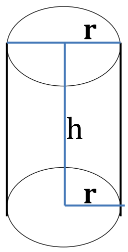
If 'r' is the radius, 'ℓ' is the length of arc and θ is of the sector then

Area = ½ ℓ r = ½ r^2θsq.u.

Perimeter = ℓ + 2r = r (θ + 2) u.

Length of the Arc 'ℓ' = rθ (θ must be in radians).

CYLINDER:



If 'r is the radius of the base of cylinder and 'h' is the height of the cylinder, then

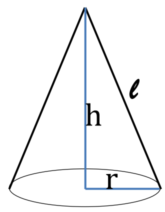
Area of base = πr^2 sq.u.

Lateral surface area = 2πrh u.

Total surface area = 2πr (h + r) u.

Volume = πr^2 h cubic units.

CONE:



If 'r' is the radius of base, 'h' is the height of cone and 'ℓ' is slant height then

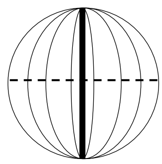
$\ell^2 + r^2 = h^2$

Lateral surface area $= \pi r \ell$ units.

Total surface area $= \pi r (\ell + r)$ sq. units.

Volume $= \frac{1}{3} \pi r^2 h$ cubic units.

SPHERE:



If 'r' is the radius of the Sphere then

Surface area $= \pi r^2$ sq. units.

Volume $= \frac{4}{3} \pi r^3$ cubic units.

11. MAXIMA AND MINIMA

Increasing and Decreasing Functions

⊗ Let f be a real function on an interval I then f is said to be

(i) an increasing function on I if
$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \ \forall \ x_1, x_2 \in I$

(ii) decreasing function on I if
$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \ \forall \ x_1, x_2 \in I$

⊗ Let f be a real function on an interval I then f is said to be

(i) strictly increasing function on I if
$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \ \forall \ x_1, x_2 \in I$

(ii) strictly decreasing function on I if
$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \ \forall \ x_1, x_2 \in I$

⊗ Let f(x) be a real valued function defined on I = (a, b) or [a, b) or (a, b] or [a, b]. Suppose f is continuous on I and differentiable in (a, b). If

(i) $f'(c) > 0 \ \forall \ c \in (a, b)$, then f is strictly increasing on I
(ii) $f'(c) < 0 \ \forall \ c \in (a, b)$, then f is strictly decreasing on I
(iii) $f'(c) \geq 0 \ \forall \ c \in (a, b)$, then f is increasing on I
(iv) $f'(c) \leq 0 \ \forall \ c \in (a, b)$, then f is decreasing on I

Critical point:

A point $x = c$ in the domain of the function said to be 'critical point' of the function f if either $f'(c) = 0$ or $f'(c)$ does not exists.

Stationary point:

A point $x = c$ in the domain of the function said to be 'stationary point' of the function f if $f'(c) = 0$.

MAXIMA & MINIMA

Global maxima – Global minima:

Let D be an interval in R and $f: D \to R$ be a real function and $c \in D$. Then f is said to be

(i) a global maximum on D if $f(c) \geq f(x)$
(ii) a global minimum on D if $f(c) \leq f(x)$

Relative maximum:

Let D be an interval in R and $f: D \to R$ be a real function and $c \in D$. Then f is said to be relative maximum at c if there exist $\delta > 0$ such that $f(c) \geq f(x)\ \forall\ x \in (c - \delta, c + \delta)$.

Here, f (c) is called relative maximum value of f(x) at $x = c$ and the point $x = c$ is called point of relative maximum.

Relative minimum:

Let D be an interval in R and $f: D \to R$ be a real function and $c \in D$. Then f is said to be relative maximum at c if there exist $\delta > 0$ such that $f(c) \leq f(x)\ \forall\ x \in (c - \delta, c + \delta)$.

Here, f (c) is called relative maximum value of f(x) at $x = c$ and the point $x = c$ is called point of relative minimum.

⊗ The relative maximum and minimum value of f are called extreme values.

⊗ If f is either minima or maxima $f'(\alpha) = 0$.

⊗ Let f be a continuous function om [a, b] and $\alpha \in (a, b)$

(i) if $f'(\alpha) = 0$ and $f''(\alpha) > 0$, then $f(\alpha)$ is relative minimum.
(ii) if $f'(\alpha) = 0$ and $f''(\alpha) < 0$, then $f(\alpha)$ is relative maximum