INVERERSE TRIGONOMETRIC FUNCTIONS

If A, B are two sets and f: A→ B is a bijection, then f^{-1} is existing and f^{-1}: B → A is an inverse function.

⊗ The function $Sin^{-1}: [-1, 1] \to [\frac{-\pi}{2}, \frac{\pi}{2}]$ is defined by $Sin^{-1} x = \theta \Leftrightarrow \theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$ and $\sin\theta = x$

⊗ The function $Cos^{-1}: [-1, 1] \to [0, \pi]$ is defined by $Sin^{-1} x = \theta \Leftrightarrow \theta \in [0, \pi]$ and $\cos\theta = x$

⊗ The function $Tan^{-1}: R \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is defined by $Tan^{-1} x = \theta \Leftrightarrow \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\theta = x$

⊗ The function $Sec^{-1}: [-\infty, -1] \cup [1, \infty] \to [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$ is defined by $Sin^{-1} x = \theta \Leftrightarrow \theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ and $\sec\theta = x$

⊗ The function $Cosec^{-1}: [-\infty, -1] \cup [1, \infty] \to [\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ is defined by $cosec^{-1} x = \theta \Leftrightarrow \theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$ and $Cosec\,\theta = x$

⊗ The function $Cot^{-1}: R \to (0, \pi)$ is defined by $Cot^{-1} x = \theta \Leftrightarrow \theta \in (0, \pi)$ and $\cot\theta = x$

Function	Domain	Range
$Sin^{-1} x$	$[-1, 1]$	$[\frac{-\pi}{2}, \frac{\pi}{2}]$
$Cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$Tan^{-1} x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$Cot^{-1} x$	R	$(0, \pi)$
$Sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$
$Cosec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

Properties of Inverse Trigonometric functions:

⊗ $Sin^{-1} x = Cosec^{-1}(1/x)\ \forall\, x \in [-1, 1] - \{0\}$

⊗ $Cos^{-1}x = Sec^{-1}(1/x)\ \forall\, x \in [-1, 1] - \{0\}$

⊗ $Tan^{-1} x = Cot^{-1}(1/x)$, if $x > 0$

⊗ $Tan^{-1} x = Cot^{-1}(1/x) - \pi$, if $x < 0$

(i) If $\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, then $Sin^{-1}(\sin\theta) = \theta$ and if $x \in [-1, 1]$, then $\sin(Sin^{-1}x) = x$

(ii) If $\theta \in [0, \pi]$, then $Cos^{-1}(\cos\theta) = \theta$ and if $x \in [-1, 1]$, then $\cos(Cos^{-1}x) = x$

(iii) If $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then $\tan^{-1}(\tan\theta) = \theta$ and if $x \in R$, then $\tan(\text{Tan}^{-1}x) = x$

(iv) If $\theta \in (0, \pi)$, then $\text{Cot}^{-1}(\cot\theta) = \theta$ and if $x \in R$, then $\cot(\text{Cot}^{-1}x) = x$

(v) If $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, then $\text{Sec}^{-1}(\sec\theta) = \theta$ and

if $x \in (-\infty, -1] \cup [1, \infty)$, then $\sec(\text{Sec}^{-1}x) = x$

(vi) If $\theta \in [\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$, then $\text{Cosec}^{-1}(\text{cosec}\,\theta) = \theta$ and

if $x \in (-\infty, -1] \cup [1, \infty)$, then $\text{cosec}(\text{Cosec}^{-1}x) = x$

(i) If $\theta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, then $\text{Cos}^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$

(ii) If $\theta \in [0, \pi]$, then $\text{Sin}^{-1}(\cos\theta) = \frac{\pi}{2} - \theta$

(iii) If $\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then $\text{Cot}^{-1}(\tan\theta) = \frac{\pi}{2} - \theta$

(iv) If $\theta \in (0, \pi)$, then $\text{Tan}^{-1}(\cot\theta) = \frac{\pi}{2} - \theta$

(v) If $\theta \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, then $\text{Cosec}^{-1}(\sec\theta) = \frac{\pi}{2} - \theta$

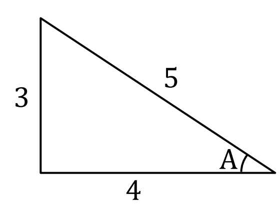
(vi) If $\theta \in [\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$, then $\text{Sec}^{-1}(\text{cosec}\,\theta) = \frac{\pi}{2} - \theta$

⊗ $\text{Tan}^{-1}x + \text{Tan}^{-1}y = \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$

⊗ $\text{Tan}^{-1}x - \text{Tan}^{-1}y = \text{Tan}^{-1}\left(\frac{x-y}{1+xy}\right)$

Solved Problems:

1. If $\sin^{-1}\left(\frac{3}{5}\right) = A$, find cos A, tan A and cot A



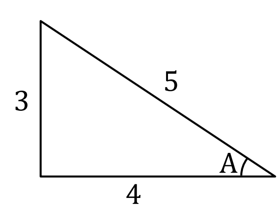
Sol: Given $\sin^{-1}\left(\frac{3}{5}\right) = A \Longrightarrow \sin A = \frac{3}{5}$

$\cos A = \frac{4}{5}$

$\tan A = \frac{3}{4}$

$\cot A = \frac{4}{3}$

2. If $\cos^{-1}\left(\frac{4}{5}\right) = A$, find sin A, tan A and cot A



Sol: Given $\cos^{-1}\left(\frac{3}{5}\right) = A \Longrightarrow \cos A = \frac{4}{5}$

$\sin A = \frac{3}{5}$

$\tan A = \frac{3}{4}$

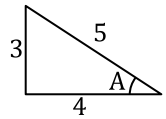
$\cot A = \frac{4}{3}$

3. Prove that $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \sin^{-1}\left(\frac{77}{85}\right)$.

Sol: Let $A = \sin^{-1}\left(\frac{3}{5}\right)$, $B = \sin^{-1}\left(\frac{8}{17}\right)$

$\Longrightarrow \sin A = \frac{3}{5}$

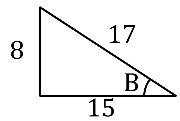
$\cos A = \frac{4}{5}$



$B = \sin^{-1}\left(\frac{8}{17}\right)$

$\Longrightarrow \sin B = \frac{8}{17}$

$\cos B = \frac{15}{17}$



$\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$= \frac{3}{5} \times \frac{15}{17} + \frac{4}{5} \times \frac{8}{17} = \frac{45}{85} + \frac{32}{85} = \frac{45 + 32}{85}$$

$\sin(A + B) = \frac{77}{85}$

$\Longrightarrow A + B = \sin^{-1}\left(\frac{77}{85}\right)$

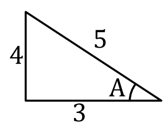
$\therefore \sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \sin^{-1}\left(\frac{77}{85}\right)$

4. Prove that $\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{16}{65}\right)$.

Sol: Let $A = \sin^{-1}\left(\frac{4}{5}\right)$, $B = \sin^{-1}\left(\frac{5}{13}\right)$

$\Longrightarrow \sin A = \frac{4}{5}$

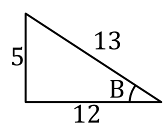
$\cos A = \frac{3}{5}$



$B = \sin^{-1}\left(\frac{5}{13}\right)$

$\Longrightarrow \sin B = \frac{5}{13}$

$\cos B = \frac{12}{13}$



$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$= \frac{3}{5} \times \frac{12}{13} + \frac{4}{5} \times \frac{5}{13} = \frac{36}{65} - \frac{20}{65} = \frac{36 - 20}{65}$$

$\cos(A + B) = \frac{16}{65}$

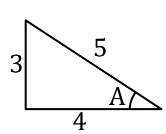
$\Longrightarrow A + B = \cos^{-1}\left(\frac{16}{65}\right)$

$\therefore \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{16}{65}\right)$

5. Prove that $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \sin^{-1}\left(\frac{56}{65}\right)$.

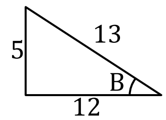
Sol: Let $A = \sin^{-1}\left(\frac{3}{5}\right)$, $B = \sin^{-1}\left(\frac{5}{13}\right)$

$\Longrightarrow \sin A = \frac{3}{5}$



$$\cos A = \frac{4}{5}$$

$$B = \sin^{-1}\left(\frac{5}{13}\right)$$



$$\Rightarrow \sin B = \frac{5}{13}$$

$$\cos B = \frac{12}{13}$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$= \frac{3}{5} \times \frac{12}{13} + \frac{4}{5} \times \frac{5}{13} = \frac{36}{65} + \frac{20}{65} = \frac{36 + 20}{65}$$

$$\sin(A + B) = \frac{56}{65}$$

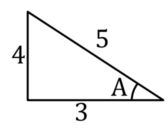
$$\Rightarrow A + B = \sin^{-1}\left(\frac{56}{65}\right)$$

$$\therefore \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \sin^{-1}\left(\frac{56}{65}\right)$$

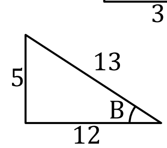
6. Prove that $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$.

Sol: Let $A = \sin^{-1}\left(\frac{4}{5}\right)$, $B = \sin^{-1}\left(\frac{5}{13}\right)$

$$\Rightarrow \sin A = \frac{3}{5}$$



$$\cos A = \frac{4}{5}$$



$$B = \sin^{-1}\left(\frac{5}{13}\right)$$

$$\Rightarrow \sin B = \frac{5}{13}$$

$$\cos B = \frac{12}{13}$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$= \frac{4}{5} \times \frac{12}{13} + \frac{3}{5} \times \frac{5}{13} = \frac{48}{65} - \frac{15}{65} = \frac{48 - 15}{65}$$

$$\cos(A + B) = \frac{33}{65}$$

$$\Rightarrow A + B = \cos^{-1}\left(\frac{33}{65}\right)$$

$$\therefore \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$$

7. Prove that $\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left(\frac{2}{9}\right)$.

Sol: Let $A = \tan^{-1}\left(\frac{1}{7}\right)$, $B = \tan^{-1}\left(\frac{1}{13}\right)$

$$\Rightarrow \tan A = \frac{1}{7}$$

$$B = \tan^{-1}\left(\frac{1}{13}\right)$$

$$\Rightarrow \tan B = \frac{1}{13}$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}.\frac{1}{13}}$$

$$= \frac{\frac{13+7}{91}}{\frac{91-1}{91}}$$

$$= \frac{\frac{20}{91}}{\frac{90}{91}} = \frac{20}{90} = \frac{2}{9}$$

$$\tan(A+B) = \frac{2}{9}$$

$$\Rightarrow A + B = \tan^{-1}\left(\frac{2}{9}\right)$$

$$\therefore \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left(\frac{2}{9}\right)$$

8. Prove that $\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{6}{17}\right)$.

Sol: Let $A = \tan^{-1}\left(\frac{1}{5}\right)$, $B = \tan^{-1}\left(\frac{1}{7}\right)$

$$\Rightarrow \tan A = \frac{1}{5}$$

$$B = \tan^{-1}\left(\frac{1}{7}\right)$$

$$\Rightarrow \tan B = \frac{1}{7}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}.\frac{1}{7}}$$

$$= \frac{\frac{7+5}{35}}{\frac{35-1}{35}}$$

$$= \frac{\frac{12}{35}}{\frac{34}{35}} = \frac{12}{34} = \frac{6}{17}$$

$$\tan(A+B) = \frac{6}{17}$$

$$\Rightarrow A + B = \tan^{-1}\left(\frac{6}{17}\right)$$

$$\therefore \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{6}{17}\right)$$

9. Prove that $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{1}{2}\right)$.

Sol: Let $A = \tan^{-1}\left(\frac{1}{3}\right)$, $B = \tan^{-1}\left(\frac{1}{7}\right)$

$$\Rightarrow \tan A = \frac{1}{3}$$

$$B = \tan^{-1}\left(\frac{1}{7}\right)$$

$$\Rightarrow \tan B = \frac{1}{7}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{1}{3}+\frac{1}{7}}{1-\frac{1}{3}.\frac{1}{7}}$$

$$= \frac{\frac{7+3}{21}}{\frac{21-1}{21}}$$

$$= \frac{\frac{10}{21}}{\frac{20}{21}} = \frac{10}{20} = \frac{1}{2}$$

$\tan(A+B) = \frac{1}{2}$

$\Longrightarrow A + B = \tan^{-1}\left(\frac{1}{2}\right)$

$\therefore \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

10. Prove that $\tan^{-1}\left(\frac{2}{7}\right) + \tan^{-1}\left(\frac{1}{4}\right) = \cot^{-1}\left(\frac{26}{15}\right)$.

Sol: Let $A = \tan^{-1}\left(\frac{2}{7}\right)$, $B = \tan^{-1}\left(\frac{26}{15}\right)$

$\Longrightarrow \tan A = \frac{2}{7}$

$B = \tan^{-1}\left(\frac{1}{4}\right)$

$\Longrightarrow \tan B = \frac{1}{4}$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{2}{7}+\frac{1}{4}}{1-\frac{2}{7}.\frac{1}{4}}$$

$$= \frac{\frac{8+7}{28}}{\frac{28-2}{28}}$$

$$= \frac{\frac{15}{28}}{\frac{26}{28}} = \frac{15}{26}$$

$\tan(A+B) = \frac{15}{26}$

$\Longrightarrow A + B = \tan^{-1}\left(\frac{15}{26}\right) = \cot^{-1}\left(\frac{26}{15}\right)$

$\therefore \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{6}{17}\right)$

11. Prove that $\tan^{-1}\left(\frac{1}{3}\right) - \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{1}{5}\right)$.

Sol: Let $A = \tan^{-1}\left(\frac{1}{3}\right)$, $B = \tan^{-1}\left(\frac{1}{8}\right)$

$\Longrightarrow \tan A = \frac{1}{3}$

$B = \tan^{-1}\left(\frac{1}{8}\right)$

$\Longrightarrow \tan B = \frac{1}{8}$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$= \frac{\frac{1}{3}-\frac{1}{8}}{1+\frac{1}{3}.\frac{1}{8}}$$

$$= \frac{\frac{8-3}{24}}{\frac{24+1}{24}}$$

$$= \frac{\frac{5}{24}}{\frac{25}{35}} = \frac{5}{25} = \frac{1}{5}$$

$$\tan(A - B) = \frac{1}{5}$$

$$\Rightarrow A - B = \tan^{-1}\left(\frac{1}{5}\right)$$

$$\therefore \tan^{-1}\left(\frac{1}{3}\right) - \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{1}{8}\right)$$

12. Prove that $\tan^{-1}\left(\frac{1}{13}\right) - \tan^{-1}\left(\frac{1}{15}\right) = \tan^{-1}\left(\frac{1}{98}\right)$.

Sol: Let $A = \tan^{-1}\left(\frac{1}{13}\right)$, $B = \tan^{-1}\left(\frac{1}{15}\right)$

$$\Rightarrow \tan A = \frac{1}{13}$$

$$B = \tan^{-1}\left(\frac{1}{15}\right)$$

$$\Rightarrow \tan B = \frac{1}{15}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$= \frac{\frac{1}{13} - \frac{1}{15}}{1 + \frac{1}{13} \cdot \frac{1}{15}}$$

$$= \frac{\frac{15-13}{195}}{\frac{195+1}{195}}$$

$$= \frac{\frac{2}{195}}{\frac{196}{195}} = \frac{2}{196} = \frac{1}{98}$$

$$\tan(A - B) = \frac{1}{98}$$

$$\Rightarrow A - B = \tan^{-1}\left(\frac{1}{98}\right)$$

$$\therefore \tan^{-1}\left(\frac{1}{13}\right) - \tan^{-1}\left(\frac{1}{15}\right) = \tan^{-1}\left(\frac{1}{98}\right)$$

13. Prove that $\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$.

Sol: Let $A = \tan^{-1}\left(\frac{3}{4}\right)$, $B = \tan^{-1}\left(\frac{1}{7}\right)$

$$\Rightarrow \tan A = \frac{3}{4}$$

$$B = \tan^{-1}\left(\frac{1}{7}\right)$$

$$\Rightarrow \tan B = \frac{1}{7}$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4} \cdot \frac{1}{7}}$$

$$= \frac{\frac{21+4}{28}}{\frac{28-3}{28}}$$

$$= \frac{\frac{25}{28}}{\frac{25}{28}} = 1$$

$\tan (A + B) = 1$

$\Longrightarrow A + B = \tan^{-1}(1) = \frac{\pi}{4}$

$\therefore \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

14. Prove that $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{4}$.

Sol: Let $A = \tan^{-1}\left(\frac{1}{4}\right)$, $B = \tan^{-1}\left(\frac{3}{5}\right)$

$\Longrightarrow \tan A = \frac{1}{4}$

$B = \tan^{-1}\left(\frac{3}{5}\right)$

$\Longrightarrow \tan B = \frac{3}{5}$

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{1}{4} + \frac{3}{5}}{1 - \frac{1}{4} \cdot \frac{3}{5}}$$

$$= \frac{\frac{5 + 12}{20}}{\frac{20 - 3}{20}}$$

$$= \frac{\frac{17}{20}}{\frac{17}{20}} = 1$$

$\tan (A + B) = 1$

$\Longrightarrow A + B = \tan^{-1}(1) = \frac{\pi}{4}$

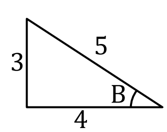
$\therefore \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

15. Prove that $\tan^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \cot^{-1}\left(\frac{11}{27}\right)$.

Sol: Let $A = \tan^{-1}\left(\frac{3}{5}\right)$, $B = \sin^{-1}\left(\frac{3}{5}\right)$

$\Longrightarrow \tan A = \frac{3}{5}$

$\Longrightarrow \cot A = \frac{5}{3}$



$B = \sin^{-1}\left(\frac{3}{5}\right)$

$\Longrightarrow \sin B = \frac{3}{5}$

$\Longrightarrow \cot B = \frac{4}{3}$

$$\cot (A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$$

$$= \frac{\frac{5}{3} \cdot \frac{4}{3} - 1}{\frac{5}{3} + \frac{4}{3}}$$

$$= \frac{\frac{20 - 9}{9}}{\frac{15 + 12}{9}}$$

$$= \frac{\frac{11}{9}}{\frac{27}{9}} = \frac{11}{27}$$

$$\cot (A + B) = \frac{11}{27}$$

$$\Longrightarrow A + B = \cot^{-1}\left(\frac{11}{27}\right)$$

$$\therefore \tan^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \cot^{-1}\left(\frac{11}{27}\right)$$

16. Prove that $\tan^{-1}\left(\frac{2}{3}\right) + \cot^{-1}\left(\frac{4}{3}\right) = \cot^{-1}\left(\frac{17}{6}\right) = \cot^{-1}\left(\frac{6}{17}\right)$.

Sol: Let $A = \tan^{-1}\left(\frac{2}{3}\right)$, $B = \cot^{-1}\left(\frac{4}{3}\right)$

$$\Longrightarrow \tan A = \frac{2}{3}$$

$$B = \cot^{-1}\left(\frac{4}{3}\right)$$

$$\Longrightarrow \cot B = \frac{4}{3}$$

$$\Longrightarrow \tan B = \frac{3}{4}$$

$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{2}{3} + \frac{3}{4}}{1 - \frac{2}{3} \cdot \frac{3}{4}}$$

$$= \frac{\frac{8+9}{12}}{\frac{12-6}{12}}$$

$$= \frac{\frac{17}{12}}{\frac{6}{12}} = \frac{17}{6}$$

$$\tan (A + B) = \frac{17}{6}$$

$$\Longrightarrow A + B = \tan^{-1}\left(\frac{17}{6}\right)$$

$$\therefore \tan^{-1}\left(\frac{2}{3}\right) - \tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{17}{6}\right) = \cot^{-1}\left(\frac{6}{17}\right)$$

17. $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{\pi}{2}$, show that $x^2 + y^2 + z^2 + 2xyz = 1$

Sol:

Let $A = \sin^{-1} x \Longrightarrow \sin A = x$

$B = \sin^{-1} y \Longrightarrow \sin B = y$

$C = \sin^{-1} z \Longrightarrow \sin C = z$

$\Longrightarrow A + B + C = 90^0$

Now

$$x^2 + y^2 + z^2 = \sin^2 A + \sin^2 B + \sin^2 C$$

$$= \frac{1 - \cos 2A}{2} + \frac{1 - \cos 2B}{2} + \frac{1 - \cos 2C}{2}$$

$$= \frac{1}{2}[1 - \cos 2A + 1 - \cos 2B + 1 - \cos 2C]$$

$$= \frac{1}{2}[3 - \{\cos 2A + \cos 2B + \cos 2C\}]$$

$= \frac{1}{2}[3 - \{2 \text{ Cos}\left(\frac{2A + 2B}{2}\right) \text{Cos}\left(\frac{2A - 2B}{2}\right) + \text{Cos } 2C\}]$

$\text{Cos C} + \text{Cos D} = 2 \text{ Cos}\left(\frac{C + D}{2}\right) \text{Cos}\left(\frac{C + D}{2}\right)$

$= \frac{1}{2}[3 - \{2\text{Cos }(A + B) \text{ Cos }(A - B) - \text{Cos } 2C\}]$

$= \frac{1}{2}[3 - \{2\text{Sin C Cos }(A - B) + 1 - 2 \text{ Sin}^2 \text{ C}\}]$

$= \frac{1}{2}[3 - \{1 + 2 \text{ Sin C }(\text{Cos }(A - B) - \text{Sin C})\}]$

$= \frac{1}{2}[3 - \{1 + 2 \text{ Sin C }(\text{Cos }(A - B) - \text{Cos }(A + B))\}]$

$= \frac{1}{2}[3 - \{1 - 2 \text{ Sin C }(2 \text{ Sin A Sin B})\}]$

$\text{Cos }(A - B) - \text{Cos }(A + B) = 2 \text{ Sin A Sin B}$

$= \frac{1}{2}[3 - \{1 - 4 \text{ Sin A Sin B Sin C}\}]$

$= \frac{1}{2}[3 - 1 + 4 \text{ Sin A Sin B Sin C}]$

$= \frac{1}{2}[2 - 4\text{Sin A Sin B Sin C}]$

$= \frac{2}{2}[1 - 2\text{Sin A Sin B Sin C}]$

$= 1 - 2\text{Sin A Sin B Sin C}$

$x^2 + y^2 + z^2 = 1 - 2xyz$

$x^2 + y^2 + z^2 + 2xyz = 1$

18. If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, then prove that $x + y + z = xyz$

Sol:

Let $A = \tan^{-1} x \Longrightarrow \tan A = x$

$B = \tan^{-1} y \Longrightarrow \tan B = y$

$C = \tan^{-1} z \Longrightarrow \tan C = z$

$\Longrightarrow A + B + C = 180^0$

$\Longrightarrow A + B = 180^0 - C$

Apply tan on both sides

$\tan (A + B) = (180^0 - C)$

$\tan (A + B) = -\tan C$

$\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$

$\text{Tan } A + \tan B = -\tan C (1 - \tan A \tan B)$

$\text{Tan } A + \tan B = -\tan C + \tan A \tan B \tan C$

$\text{Tan } A + \tan B + \tan C = \tan A \tan B \tan C$

$\therefore x + y + z = xyz$

19. If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$, then prove that $xy + yz + zx = 1$

Sol:

Let $A = \tan^{-1} x \Longrightarrow \tan A = x$

$B = \tan^{-1} y \Longrightarrow \tan B = y$

$C = \tan^{-1} z \Longrightarrow \tan C = z$

$\Longrightarrow A + B + C = 90^0$

$\Longrightarrow A + B = 90^0 - C$

Apply tan on both sides

$\tan (A + B) = \tan (90^0 - C)$

$\tan (A + B) = \cot C$

$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{1}{\tan C}$

$\tan C \tan A + \tan C \tan B = 1 - \tan A \tan B$

$\tan C \tan A + \tan B \tan C + \tan A \tan B = 1$

$\therefore xy + yz + zx = 1$

20. If $\cot^{-1} x + \cot^{-1} y + \cot^{-1} z = \frac{\pi}{2}$, then prove that $x + y + z = xyz$.

Sol:

Let $A = \cot^{-1} x \Longrightarrow \cot A = x$

$B = \cot^{-1} y \Longrightarrow \cot B = y$

$C = \cot^{-1} z \Longrightarrow \cot C = z$

$\Longrightarrow A + B + C = 90^0$

$\Longrightarrow A + B = 90^0 - C$

Apply tan on both sides

$\cot (A + B) = \cot (90^0 - C)$

$\cot (A + B) = \tan C$

$\frac{\cot A \cot B - 1}{\cot A + \cot B} = \frac{1}{\cot C}$

$\cot C (\cot A \cot B - 1) = \cot A + \cot B$

$\cot A \cot B \cot C - \cot C = \cot A + \cot B$

$\cot A \cot B \cot C = \cot A + \cot B + \cot C$

$\therefore x + y + z = xyz$

21. $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, show that $x^2 + y^2 + z^2 + 2xyz = 1$

Sol:

Let $A = \cos^{-1} x \Longrightarrow \cos A = x$

$B = \cos^{-1} y \Longrightarrow \cos B = y$

$C = \cos^{-1} z \Longrightarrow \cos C = z$

$\Longrightarrow A + B + C = 180^0$

Now

$x^2 + y^2 + z^2 = \cos^2 A + \cos^2 B + \cos^2 C$

$= \frac{1 + \text{Cos } 2A}{2} + \frac{1 + \text{Cos } 2B}{2} + \frac{1 + \text{Cos } 2C}{2}$

$= \frac{1}{2}[1 + \text{Cos } 2A + 1 + \text{Cos } 2B + 1 + \text{Cos } 2C]$

$= \frac{1}{2}[3 + \{\text{Cos } 2A + \text{Cos } 2B + \text{Cos } 2C\}]$

$= \frac{1}{2}[3 + \{2 \text{ Cos} \left(\frac{2A + 2B}{2}\right) \text{Cos} \left(\frac{2A - 2B}{2}\right) + \text{Cos } 2C\}]$

$\text{Cos C} + \text{Cos D} = 2 \text{ Cos}\left(\frac{C + D}{2}\right) \text{Cos} \left(\frac{C + D}{2}\right)$

$= \frac{1}{2}[3 + \{2\text{Cos } (A + B) \text{ Cos } (A - B) - \text{Cos } 2C\}]$

$= \frac{1}{2}[3 + \{- 2\text{Cos C Cos } (A - B) + 2 \text{ Cos}^2 \text{ C} - 1\}]$

$= \frac{1}{2}[3 + \{- 1 - 2 \text{ Cos C } (\text{Cos } (A - B) - \text{Cos C})\}]$

$= \frac{1}{2}[3 + \{- 1 - 2 \text{ Cos C } (\text{Cos } (A - B) + \text{Cos } (A + B))\}]$

$= \frac{1}{2}[3 + \{- 1 - 2 \text{ Cos C } (2 \text{ Cos A Cos B})\}]$

$\text{Cos } (A - B) + \text{Cos } (A + B) = 2 \text{ Cos A Cos B}$

$= \frac{1}{2}[3 + \{- 1 - 4 \text{ Cos A Cos B Cos C}\}]$

$= \frac{1}{2}[3 - 1 + 4 \text{ Cos A Cos B Cos C}]$

$= \frac{1}{2}[2 - 4 \text{ Cos A Cos B Cos C}]$

$= \frac{2}{2}[1 - 2 \text{ Cos A Cos B Cos C}]$

$= 1 - 2 \text{ Cos A Cos B Cos C}$

$x^2 + y^2 + z^2 = 1 - 2xyz$

$\therefore x^2 + y^2 + z^2 + 2xyz = 1$

22. Solve $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

Sol:

Let $A = \tan^{-1} 2x \Rightarrow \tan A = 2x$

$B = \tan^{-1} 3x \Rightarrow \tan B = 3x$

$\Rightarrow A + B = 45^0$

Apply tan on both sides

$\tan (A + B) = \tan 45^0$

$\tan (A + B) = 1$

$\frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$

$\frac{2x + 3x}{1 - 2x \cdot 3x} = 1$

$5x = 1 - 6x^2$

$6x^2 + 5x - 1 = 0$

$6x^2 + 6x - x - 1 = 0$

$6x (x + 1) - 1(x + 1) = 0$

$(x + 1) (6x - 1) = 0$

$x = -1$ or $x = \frac{1}{6}$

$x = -1$ not satisfy the equation

$\therefore x = \frac{1}{6}$

23. Solve $\tan^{-1}(1+x) + \tan^{-1}(1-x) = \tan^{-1}\frac{1}{2}$

Sol:

Let $A = \tan^{-1}(1+x) \Rightarrow \tan A = 1 + x$

$B = \tan^{-1}(1-x) \Rightarrow \tan B = 1 - x$

$C = \tan^{-1}\frac{1}{2} \Rightarrow \tan C = \frac{1}{2}$

$\Rightarrow A + B = C$

Apply tan on both sides

$\tan(A+B) = \tan C$

$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{1}{2}$

$\frac{1 + x + 1 - x}{1-(1+x)(1-x)} = \frac{1}{2}$

$\frac{2}{1-(1-x^2)} = \frac{1}{2}$

$\frac{2}{x^2} = \frac{1}{2}$

$x^2 = 4$

$\therefore x = \pm 2$

24. Solve $\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}$.

Sol: Let $A = \sin^{-1}\left(\frac{5}{x}\right)$, $B = \sin^{-1}\left(\frac{12}{x}\right)$

$\Rightarrow \sin A = \frac{5}{x}$

$B = \sin^{-1}\left(\frac{12}{x}\right)$

$\Rightarrow \sin B = \frac{12}{x}$

$A + B = 90^0$

$\Rightarrow A = 90^0 - B$

$\sin A = \sin(90^0 - B) = \cos B = \sqrt{1 - \sin^2 A}$

$\frac{5}{x} = \sqrt{1 - \left(\frac{12}{x}\right)^2}$

$\frac{5}{x} = \sqrt{1 - \frac{144}{x^2}}$

$\frac{25}{x^2} = 1 - \frac{144}{x^2}$

$\frac{25}{x^2} = \frac{x^2 - 144}{x^2}$

$25 = x^2 - 144$

$x^2 = 144 + 25 = 169$

$x = \pm 13$

$x = -13$ does not satisfy the equation

$x = 13$ is the solution

25. If $\cos^{-1}\left(\frac{x}{2}\right) + \cos^{-1}\left(\frac{y}{3}\right) = \theta$. Prove that $9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$

Sol: Let $A = \cos^{-1}\left(\frac{x}{2}\right)$, $B = \cos^{-1}\left(\frac{y}{3}\right)$

$\Longrightarrow \text{Cos } A = \frac{x}{2}$

$B = \cos^{-1}\left(\frac{y}{3}\right)$

$\Longrightarrow \cos B = \frac{y}{3}$

$A + B = \theta$

$\cos(A + B) = \cos\theta$

$\cos A \cos B - \sin A \sin B = \cos\theta$

$\frac{x}{2} \cdot \frac{y}{3} - \sqrt{1 - \cos^2 A}\,\sqrt{1 - \cos^2 B} = \cos\theta$

$\frac{xy}{6} - \sqrt{1 - \left(\frac{x}{2}\right)^2}\ \sqrt{1 - \left(\frac{y}{3}\right)^2} = \cos\theta$

$\frac{xy}{6} - \sqrt{1 - \frac{x^2}{4}}\ \sqrt{1 - \frac{y^2}{9}} = \cos\theta$

$-\sqrt{1 - \frac{x^2}{4}}\ \sqrt{1 - \frac{y^2}{9}} = \cos\theta - \frac{xy}{6}$

Squaring on both sides

$\left(-\sqrt{1 - \frac{x^2}{4}}\ \sqrt{1 - \frac{y^2}{9}}\right)^2 = \left(\cos\theta - \frac{xy}{6}\right)^2$

$\left(1 - \frac{x^2}{4}\right)\left(1 - \frac{y^2}{9}\right) = \cos^2\theta + \left(\frac{xy}{6}\right)^2 - 2\cos\theta\,\frac{xy}{6}$

$1 - \frac{x^2}{4} - \frac{y^2}{9} + \frac{x^2y^2}{36} = \cos^2\theta + \frac{x^2y^2}{36} - \cos\theta \cdot \frac{xy}{3}$

$1 - \frac{x^2}{4} - \frac{y^2}{9} = \cos^2\theta - \cos\theta \cdot \frac{xy}{3}$

$1 - \cos^2\theta = \frac{x^2}{4} + \frac{y^2}{9} - \cos\theta \cdot \frac{xy}{3}$

$\text{Sin}^2\theta = \frac{9x^2 + 4y^2 - 12xy\cos\theta}{36}$

$36\sin^2\theta = 9x^2 + 4y^2 - 12xy\cos\theta$

$\therefore 9x^2 - 12xy\cos\theta + 4y^2 = 36\sin^2\theta$