SOLUTIONS OF TRIANGLES

Sine rule:

In ΔABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$\Longrightarrow a = 2R \sin A;\ b = 2R \sin B;\ c = 2R \sin C$

Where R is the circum radius and a, b, c, are lengths of the sides of ΔABC.

Cosine rule:

In ΔABC,

$a^2 = b^2 + c^2 - 2bc \cos A \quad \Longrightarrow \cos A = \frac{b^2+c^2-a^2}{2bc}$

$b^2 = a^2 + c^2 - 2ac \cos B \quad \Longrightarrow \cos B = \frac{a^2+c^2-b^2}{2ac}$

$c^2 = a^2 + b^2 - 2ab \cos C \quad \Longrightarrow \cos A = \frac{a^2+b^2-a^2}{2ab}$

Solved problems

1. Solve the triangle ABC, given $a = 2$, $b = \sqrt{2}$ and $c = \sqrt{3} + 1$

   Sol: Given $a = 2$, $b = \sqrt{2}$, $c = \sqrt{3} + 1$

   $$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(\sqrt{2})^2 + (\sqrt{3}+1)^2 - (2)^2}{2(\sqrt{2})(\sqrt{3}+1)} = \frac{2+3+1+2\sqrt{3}-4}{2(\sqrt{2})(\sqrt{3}+1)} = \frac{2+2\sqrt{3}}{2(\sqrt{2})(\sqrt{3}+1)} = \frac{2(1+\sqrt{3})}{2(\sqrt{2})(\sqrt{3}+1)}$$

   $\Longrightarrow \cos A = \frac{1}{\sqrt{2}} \Longrightarrow A = 45^0$

   $$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{(2)^2 + (\sqrt{3}+1)^2 - (\sqrt{2})^2}{2(2)(\sqrt{3}+1)} = \frac{4+3+1+2\sqrt{3}-2}{2(2)(\sqrt{3}+1)} = \frac{6+2\sqrt{3}}{2(2)(\sqrt{3}+1)} = \frac{2(3+\sqrt{3})}{2(2)(\sqrt{3}+1)}$$

   $$= \frac{(3+\sqrt{3})}{2(\sqrt{3}+1)} = \frac{\sqrt{3}\,(1+\sqrt{3})}{2(\sqrt{3}+1)}$$

   $\Longrightarrow \cos B = \frac{\sqrt{3}}{2} \Longrightarrow B = 30^0$

   We know that $A + B + C = 180^0$

   $\Longrightarrow C = 180^0 - (A + B)$

   $C = 180^0 - (30^0 + 45^0)$

   $C = 180^0 - 75^0 = 105^0$

   $\therefore A = 45^0$, $B = 30^0$ and $C = 105^0$

2. Solve the triangle ABC, given $a = 1$, $b = 2$ and $c = \sqrt{3}$

   Sol: Given $a = 1$, $b = 2$, $c = \sqrt{3}$

   $$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{(2)^2 + (\sqrt{3})^2 - (1)^2}{2(2)(\sqrt{3})} = \frac{4+3-1}{4\sqrt{3}} = \frac{6}{4\sqrt{3}} = \frac{3}{2\sqrt{3}}$$

$\Longrightarrow \cos A = \frac{\sqrt{3}}{2} \Longrightarrow A = 30^0$

$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{(1)^2+(\sqrt{3})^2-(2)^2}{2(1)(\sqrt{3})} = \frac{1+3-4}{2(\sqrt{3})} = \frac{0}{2\sqrt{3}}$

$\cos B = 0 \Longrightarrow B = 90^0$

We know that $A + B + C = 180^0$

$\Longrightarrow C = 180^0 - (A + B)$

$C = 180^0 - (30^0 + 90^0)$

$C = 180^0 - 120^0 = 60^0$

$\therefore A = 30^0, B = 90^0$ and $C = 120^0$

3. Solve the triangle ABC, given $a = 1, b = \sqrt{3}$ and $c = 2$

Sol: Given $a = 1, b = \sqrt{3}, c = 2$

$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{(\sqrt{3})^2+(2)^2-(1)^2}{2(\sqrt{3})(2)} = \frac{3+4-1}{4\sqrt{3}} = \frac{6}{4\sqrt{3}} = \frac{3}{2\sqrt{3}}$

$\Longrightarrow \cos A = \frac{\sqrt{3}}{2} \Longrightarrow A = 30^0$

$\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{(1)^2+(2)^2-(\sqrt{3})^2}{2(1)(2)} = \frac{1+4-3}{4} = \frac{2}{4}$

$\cos B = \frac{1}{2} \Longrightarrow B = 60^0$

We know that $A + B + C = 180^0$

$\Longrightarrow C = 180^0 - (A + B)$

$C = 180^0 - (30^0 + 60^0)$

$C = 180^0 - 90^0 = 90^0$

$\therefore A = 30^0, B = 60^0$ and $C = 60^0$

4. Solve the triangle ABC, given $a = 2, A = 30^0$ and $C = 30^0$

Sol: Given $a = 1, A = 30^0, C = 30^0$

We know that $A + B + C = 180^0$

$30^0 + B + 30^0 = 180^0$

$B = 180^0 - 60^0 = 120^0$

From 'sin' Rule

$\frac{a}{\sin A} = \frac{b}{\sin B}$

$\Longrightarrow a \sin B = b \sin A$

$2 \cdot \sin 120^0 = b \sin 30^0$

$2\sin(180^0 - 60^0) = b \cdot \frac{1}{2}$

$4\sin 60^0 = b \Longrightarrow 4\frac{\sqrt{3}}{2} = b$

$b = 2\sqrt{3}$

$\frac{a}{\sin A} = \frac{c}{\sin C} \Longrightarrow a \sin C = c \sin A$

$2 \cdot \sin 30^0 = c \sin 30^0$

$2 \cdot \frac{1}{2} = c \cdot \frac{1}{2}$

$c = 2$

$\therefore b = 2\sqrt{3}, c = 2$ and $B = 120^0$

5. Solve the triangle ABC, given $b = 1, A = 30^0$ and $c = \sqrt{3}$

Sol: Given $a = 1, A = 30^0, c = \sqrt{3}$

We know that $a^2 = b^2 + c^2 - 2bc \cos A$

$a^2 = 1^2 + (\sqrt{3})^2 - 2b\sqrt{3} \cos 30^0$

$a^2 = 1 + 3 - 2\sqrt{3}\frac{\sqrt{3}}{2}$

$a = 1$

From 'sin' Rule

$\frac{a}{\sin A} = \frac{b}{\sin B}$

$a \sin B = b \text{ in } A$

$\Longrightarrow 1 \sin B = 1 \sin 30^0$

$\sin B = \sin 30^0$

$B = 30^0$

We know that $A + B + C = 180^0$

$30^0 + 30^0 + C = 180^0$

$C = 180^0 - 60^0 = 120^0$

$\therefore a = 1, B = 30^0$ and $C = 120^0$

6. Solve the triangle ABC, given $b = \sqrt{8}, c = \sqrt{12}$ and $B = 45^0$

Sol: Given $b = \sqrt{8}, c = \sqrt{12}, B = 45^0$

From 'sin' Rule

$\frac{b}{\sin B} = \frac{c}{\sin C}$

$\Longrightarrow b \sin C = c \sin B$

$\Longrightarrow \sqrt{8} \sin C = \sqrt{12} \sin 45^0$

$\sqrt{8} \sin C = \sqrt{12}\frac{1}{\sqrt{2}}$

$\sqrt{16} \sin C = \sqrt{12}$

$\text{Sin } C = \frac{\sqrt{12}}{\sqrt{16}} = \frac{2\sqrt{3}}{4}$

$\text{Sin } C = \frac{\sqrt{3}}{2}$

$C = 60^0$

We know that $A + B + C = 180^0$

$A + 45^0 + 60^0 = 180^0$

$A = 180^0 - 105^0 = 75^0$

$\frac{a}{\sin A} = \frac{b}{\sin B}$

$a \sin B = b \sin 75^0$

$a \sin 45^0 = \sqrt{8} \sin 75^0$

$a \frac{1}{\sqrt{2}} = \sqrt{8}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$

$a = \sqrt{2}\,(\sqrt{3}+1) = \sqrt{6} + \sqrt{2}$

$\therefore a = \sqrt{6} + \sqrt{2}$, $A = 75^0$ and $C = 60^0$

7. Solve the triangle ABC, given $b = \sqrt{8}$, $c = \sqrt{12}$ and $B = 45^0$

Sol: Given $b = \sqrt{8}$, $c = \sqrt{12}$, $B = 45^0$

From 'sin' Rule

$\frac{b}{\sin B} = \frac{c}{\sin C}$

$\Longrightarrow b \sin C = c \sin B$

$\Longrightarrow \sqrt{8} \sin C = \sqrt{12} \sin 45^0$

$\sqrt{8} \sin C = \sqrt{12}\,\frac{1}{\sqrt{2}}$

$\sqrt{16} \sin C = \sqrt{12}$

$\text{Sin } C = \frac{\sqrt{12}}{\sqrt{16}} = \frac{2\sqrt{3}}{4}$

$\text{Sin } C = \frac{\sqrt{3}}{2}$

$C = 60^0$

We know that $A + B + C = 180^0$

$A + 45^0 + 60^0 = 180^0$

$A = 180^0 - 105^0 = 75^0$

$\frac{a}{\sin A} = \frac{b}{\sin B}$

$a \sin B = b \sin 75^0$

$a \sin 45^0 = \sqrt{8} \sin 75^0$

$a \frac{1}{\sqrt{2}} = \sqrt{8}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$

$a = \sqrt{2}\,(\sqrt{3}+1) = \sqrt{6} + \sqrt{2}$

$\therefore a = \sqrt{6} + \sqrt{2}$, $A = 75^0$ and $C = 60^0$

8. Solve the triangle ABC, given $b = \sqrt{3}$, $c = 1$ and $A = 30^0$

Sol: Given $b = \sqrt{3}$, $c = 1$, $B = 30^0$

We know that $a^2 = b^2 + c^2 - 2bc \cos A$

$a^2 = 1^2 + (\sqrt{3})^2 - 2b\sqrt{3} \cos 30^0$

$a^2 = 1 + 3 - 2\sqrt{3}\,\frac{\sqrt{3}}{2}$

$a = 1$

From 'sin' Rule

$\frac{a}{\sin A} = \frac{c}{\sin C}$

$a \sin C = b \text{ in } A$

$\Longrightarrow 1 \sin C = 1 \sin 30^0$

$\sin C = \sin 30^0$

$C = 30^0$

We know that $A + B + C = 180^0$

$30^0 + B + 30^0 = 180^0$

$B = 180^0 - 60^0 = 120^0$

$\therefore a = 1, B = 120^0$ and $C = 30^0$