TRANSFORMATIONS – PART 1

For A, B∈ R

- ⁎ $\sin(A+B)+\sin(A-B)=2\sin A\cos B$
- ⁎ $\sin(A+B)-\sin(A-B)=2\cos A\sin B$
- ⁎ $\cos(A+B)+\cos(A-B)=2\cos A\cos B$
- ⁎ $\cos(A+B)-\cos(A-B)=-2\sin A\sin B$

For any two real numbers C and D

- ⁎ $\sin C+\sin D=2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- ⁎ $\sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
- ⁎ $\cos C+\cos D=2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- ⁎ $\cos C-\cos D=-2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

If $A+B+C=\pi$ or 180^0, then

- ⁎ $\sin(A+B)=\sin C$; $\sin(B+C)=\sin A$; $\sin(A+C)=\sin B$
- ⁎ $\cos(A+B)=-\cos C$; $\cos(B+C)=-\cos A$; $\cos(A+C)=-\cos B$

If $A+B+C=90^0$ or $\frac{\pi}{2}$ then

- ⁎ $\sin\left(\frac{A+B}{2}\right)=\cos\frac{C}{2}$; $\sin\left(\frac{B+C}{2}\right)=\cos\frac{A}{2}$; $\sin\left(\frac{A+C}{2}\right)=\cos\frac{B}{2}$
- ⁎ $\cos\left(\frac{A+B}{2}\right)=\sin\frac{C}{2}$; $\cos\left(\frac{B+C}{2}\right)=\sin\frac{A}{2}$; $\cos\left(\frac{A+C}{2}\right)=\sin\frac{B}{2}$

Solved Problems:

1. Convert Sin 14A – Sin 6A into product.

Sol:

$$\text{Sin } 14A-\text{Sin } 6A=2\cos\left(\frac{14A+6A}{2}\right)\sin\left(\frac{14A-6A}{2}\right)$$
$$=2\cos\left(\frac{20A}{2}\right)\sin\left(\frac{8A}{2}\right)$$
$$=2\cos 10A\sin 4A$$

2. Convert Sin 5θ + Sin 3 θ into product.

Sol:

$$\text{Sin } 5\theta+\text{Sin } 3\theta=2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)$$
$$=2\sin\left(\frac{8\theta}{2}\right)\cos\left(\frac{2\theta}{2}\right)$$
$$=2\sin 4\theta\cos\theta$$

3. Convert Sin 5A – Sin 63 into product.

Sol:

$$\text{Sin } 5A-\text{Sin } 3A=2\cos\left(\frac{5A+3A}{2}\right)\sin\left(\frac{5A-3A}{2}\right)$$
$$=2\cos\left(\frac{8A}{2}\right)\sin\left(\frac{2A}{2}\right)$$
$$=2\cos 4\sin A$$

4. Convert Sin 75^0 – Sin 15^0 into product.

Sol:

$$\text{Sin } 75^0 - \text{Sin } 15^0 = 2\cos\left(\frac{75+15}{2}\right)\sin\left(\frac{75-15}{2}\right)$$
$$= 2\cos\left(\frac{90}{2}\right)\sin\left(\frac{60}{2}\right)$$
$$= 2\cos 45^0 \sin 30^0$$

5. Convert cos 15A + cos 5A into product.

Sol:

$$\text{Sin } 75^0 - \text{Sin } 15^0 = 2\cos\left(\frac{75+15}{2}\right)\sin\left(\frac{75-15}{2}\right)$$
$$= 2\cos\left(\frac{90}{2}\right)\sin\left(\frac{60}{2}\right)$$
$$= 2\cos 45^0 \sin 30^0$$

6. Convert cos 7A – cos 17A into product.

Sol:

$$\text{Cos } 7A - \cos 17A = -2\sin\left(\frac{7A+17A}{2}\right)\sin\left(\frac{7A-17A}{2}\right)$$
$$= -2\sin\left(\frac{24A}{2}\right)\sin\left(\frac{-10A}{2}\right)$$
$$= -2\sin 12A \sin(-5A)$$
$$= 2\sin 12A \sin 5A$$

7. Show that $\frac{\cos 3A - \cos 5A}{\sin 3A + \sin 5A} = \tan A$

Sol:

$$\frac{\cos 3A - \cos 5A}{\sin 3A + \sin 5A} = \frac{-2\sin\left(\frac{3A+5A}{2}\right)\sin\left(\frac{3A-5A}{2}\right)}{2\sin\left(\frac{3A+5\theta}{2}\right)\cos\left(\frac{3A-5A}{2}\right)}$$
$$= \frac{-2\sin\left(\frac{8A}{2}\right)\sin\left(\frac{-2A}{2}\right)}{2\sin\left(\frac{8A}{2}\right)\cos\left(\frac{-2A}{2}\right)}$$
$$= \frac{2\sin 4A \sin A}{2\sin 4A \cos A} = \frac{\sin A}{\cos A} = \tan A$$

8. Show that $\frac{\sin 17A + \sin 7A}{\cos 17A + \cos 7A} = \tan A$

Sol:

$$\frac{\sin 17A + \sin 7A}{\cos 17A + \cos 7A} = \frac{2\sin\left(\frac{17A+7A}{2}\right)\cos\left(\frac{17A-7A}{2}\right)}{2\cos\left(\frac{17A+7A}{2}\right)\cos\left(\frac{17A-7A}{2}\right)}$$
$$= \frac{2\sin\left(\frac{24A}{2}\right)\cos\left(\frac{10A}{2}\right)}{2\cos\left(\frac{24A}{2}\right)\cos\left(\frac{10A}{2}\right)}$$
$$= \frac{2\sin 12A \cos 5A}{2\cos 12A \cos 5A} = \frac{\sin 12A}{\cos 12A} = \tan 12A$$

9. Prove that $\frac{\cos 3A + \cos A}{\sin 3A + \sin A} = \cot 2A$

Sol:

$$\frac{\cos 3A + \cos A}{\sin 3A + \sin A} = \frac{2\cos\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)}{2\sin\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right)}$$

$$= \frac{2\cos\left(\frac{4A}{2}\right)\cos\left(\frac{2A}{2}\right)}{2\sin\left(\frac{4A}{2}\right)\cos\left(\frac{2A}{2}\right)}$$

$$= \frac{2\cos 2A\cos A}{2\sin 2A\cos A} = \frac{\cos 2A}{\sin 2A} = \cot 2A$$

10. Prove that $\frac{\cos 7A + \cos 5A}{\sin 7A + \sin 5A} = \cot 6A$

Sol:

$$\frac{\cos 7A + \cos 5A}{\sin 7A + \sin 5A} = \frac{2\cos\left(\frac{7A+5A}{2}\right)\cos\left(\frac{7A-5A}{2}\right)}{2\sin\left(\frac{7A+5A}{2}\right)\cos\left(\frac{7A-5A}{2}\right)}$$

$$= \frac{2\cos\left(\frac{12A}{2}\right)\cos\left(\frac{2A}{2}\right)}{2\sin\left(\frac{12A}{2}\right)\cos\left(\frac{2A}{2}\right)}$$

$$= \frac{2\cos 6A\cos A}{2\sin 6A\cos A} = \frac{\cos 6A}{\sin 6A} = \cot 6A$$

11. Show that $\frac{\sin 8A + \sin 6A}{\cos 8A + \cos 6A} = \tan 7A$

Sol:

$$\frac{\sin 8A + \sin 6A}{\cos 8A + \cos 6A} = \frac{2\sin\left(\frac{8A+6A}{2}\right)\cos\left(\frac{8A-6A}{2}\right)}{2\cos\left(\frac{8A+6A}{2}\right)\cos\left(\frac{8A-6A}{2}\right)}$$

$$= \frac{2\sin\left(\frac{14A}{2}\right)\cos\left(\frac{2A}{2}\right)}{2\cos\left(\frac{14A}{2}\right)\cos\left(\frac{2A}{2}\right)}$$

$$= \frac{2\sin 7A\cos A}{2\cos 7A\cos A} = \frac{\sin 7A}{\cos 7A} = \tan 7A$$

12. Show that $\frac{\cos 27^0 + \sin 27^0}{\cos 27^0 - \sin 27^0} = \cot 18^0$

Sol:

$$\frac{\cos 27^0 + \sin 27^0}{\cos 27^0 - \sin 27^0} = \frac{\cos(90^0 - 63^0) + \sin 27^0}{\cos(90^0 - 63^0) - \sin 27^0}$$

$$= \frac{\sin 63^0 + \sin 27^0}{\sin 63^0 - \sin 27^0}$$

$$= \frac{2\sin\left(\frac{63^0 + 27^0}{2}\right)\cos\left(\frac{63^0 - 27^0}{2}\right)}{2\cos\left(\frac{63^0 + 27^0}{2}\right)\sin\left(\frac{63^0 - 27^0}{2}\right)}$$

$$=\frac{2\sin\left(\frac{90^0}{2}\right)\cos\left(\frac{36^0}{2}\right)}{2\cos\left(\frac{90^0}{2}\right)\sin\left(\frac{36^0}{2}\right)}$$

$$=\frac{2\sin 45^0\ \ A\ \cos 18^0}{2\cos 45^0\ \ A\ \sin 18^0} = \frac{2\times\frac{1}{\sqrt{2}}\times\cos 18^0}{2\times\frac{1}{\sqrt{2}}\times\sin 18^0} = \cot 18^0$$

13. Prove that $\cos(2x - 3y) - \cos(3x - 2y) = 2\sin\left(\frac{5x-5y}{2}\right)\sin\left(\frac{x+y}{2}\right)$

Sol:

$$\cos(2x - 3y) - \cos(3x - 2y) = -2\sin\left(\frac{2x-3y+3x-2y}{2}\right)\sin\left(\frac{2x-3y-3x+2y}{2}\right)$$

$$= -2\sin\left(\frac{5x-5y}{2}\right)\sin\left(\frac{-x-y}{2}\right)$$

$$= 2\sin\left(\frac{5x-5y}{2}\right)\sin\left(\frac{x+y}{2}\right)$$

14. Prove that $\frac{\sin^2 A - \sin^2 B}{\sin A\cos A - \sin B\cos B} = \tan(A+B)$

Sol:

$$\frac{\sin^2 A - \sin^2 B}{\sin A\cos A - \sin B\cos B} = \frac{\sin(A+B)\sin(A-B)}{\frac{1}{2}[2\sin A\cos A - 2\sin B\cos B]}$$

$$= \frac{2\sin(A+B)\sin(A-B)}{\sin 2A - \sin 2B}$$

$$= \frac{2\sin(A+B)\sin(A-B)}{2\cos\left(\frac{2A+2B}{2}\right)\sin\left(\frac{2A-2B}{2}\right)}$$

$$= \frac{2\sin(A+B)\sin(A-B)}{2\cos(A+B)\sin(A-B)} = \tan(A+B)$$

15. Prove that $\frac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A$

Sol:

$$\frac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A}$$

$$= \frac{\sin 7A + \sin A + \sin 5A + \sin 3A}{\cos 7A + \cos A + \cos 5A + \cos 3A}$$

$$= \frac{2\sin\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right) + 2\sin\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}{2\cos\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right) + 2\cos\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}$$

$$= \frac{2\sin\left(\frac{8A}{2}\right)\cos\left(\frac{6A}{2}\right) + 2\sin\left(\frac{8A}{2}\right)\cos\left(\frac{2A}{2}\right)}{2\cos\left(\frac{8A}{2}\right)\cos\left(\frac{6A}{2}\right) + \cos\left(\frac{8A}{2}\right)\cos\left(\frac{2A}{2}\right)}$$

$$= \frac{2\sin 4A\cos 3A + \sin 4A\cos A}{2\cos 4A\cos 3A + \cos 4A\cos A}$$

$$= \frac{2\sin 4A\,(\cos 3A + \cos A)}{2\cos 4A\,(\cos 3A + \cos A}$$

$$= \frac{\sin 4A}{\cos 4A}$$

$$= \tan 4A$$

16. Prove that $\frac{\sin A + \sin 2A + \sin 3A + \sin 4A}{\cos A + \cos 2A - \cos 3A - \cos 4A} = \cot A$

Sol:

$$\frac{\sin A + \sin 2A + \sin 3A + \sin 4A}{\cos A + \cos 2A - \cos 3A - \cos 4A}$$

$$= \frac{\sin 3A + \sin A + \sin 4A + \sin 2A}{\cos A - \cos 3A + \cos 2A - \cos 4A}$$

$$= \frac{2\sin\left(\frac{3A+A}{2}\right)\cos\left(\frac{3A-A}{2}\right) + 2\sin\left(\frac{4A+2A}{2}\right)\cos\left(\frac{4A-2A}{2}\right)}{-2\sin\left(\frac{A+3A}{2}\right)\sin\left(\frac{A-3A}{2}\right) - 2\sin\left(\frac{2A+4A}{2}\right)\sin\left(\frac{2A-4A}{2}\right)}$$

$$= \frac{2\sin\left(\frac{4A}{2}\right)\cos\left(\frac{2A}{2}\right) + 2\sin\left(\frac{6A}{2}\right)\cos\left(\frac{2A}{2}\right)}{-2\sin\left(\frac{4A}{2}\right)\sin\left(\frac{-2A}{2}\right) - \sin\left(\frac{6A}{2}\right)\sin\left(\frac{-2A}{2}\right)}$$

$$= \frac{2\sin 2A\cos A + \sin 3A\cos A}{-2\sin 2A\sin(-A) - \sin 3A\sin(-A)}$$

$$= \frac{2\cos A\,(\sin 2A + \sin 3A)}{2\sin A\,(\sin 2A + \sin 3A)}$$

$$= \frac{\cos A}{\sin A}$$

$$= \cot A$$

17. Prove that $\frac{\cos 7A + \cos 5A + \cos 3A + \cos A}{\sin 7A + \sin 5A + \sin 3A + \sin A} = \cot 4A$

Sol:

$$\frac{\cos 7A + \cos 5A + \cos 3A + \cos A}{\sin 7A + \sin 5A + \sin 3A + \sin A}$$

$$= \frac{\cos 7A + \cos A + \cos 5A + \cos 3A}{\sin 7A + \sin A + \sin 5A + \sin 3A}$$

$$= \frac{2\cos\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right) + 2\cos\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}{2\sin\left(\frac{7A+A}{2}\right)\cos\left(\frac{7A-A}{2}\right) + 2\sin\left(\frac{5A+3A}{2}\right)\cos\left(\frac{5A-3A}{2}\right)}$$

$$= \frac{2\cos\left(\frac{8A}{2}\right)\cos\left(\frac{6A}{2}\right) + 2\cos\left(\frac{8A}{2}\right)\cos\left(\frac{2A}{2}\right)}{2\sin\left(\frac{8A}{2}\right)\cos\left(\frac{6A}{2}\right) + \sin\left(\frac{8A}{2}\right)\cos\left(\frac{2A}{2}\right)}$$

$$= \frac{2\cos 4A \cos 3A + \cos 4A \cos A}{2\sin 4A \cos 3A + \sin 4A \cos A}$$

$$= \frac{2\cos 4A\,(\cos 3A + \cos A)}{2\sin 4A\,(\cos 3A + \cos A}$$

$$= \frac{\cos 4A}{\sin 4A}$$

$$= \cot 4A$$

18. Prove that $\frac{\sin 3A\ \sin 7A + \sin A \sin 11A}{\sin 3A \cos 7A + \sin A \cos 11A} = \tan 8A$

Sol: $\frac{\sin 3A\ \sin 7A + \sin A \sin 11A}{\sin 3A \cos 7A + \sin A \cos 11A}$

$$= \frac{\frac{1}{2}[2\sin 3A\ \sin 7A + 2\sin A \sin 11A]}{\frac{1}{2}[2\sin 3A \cos 7A + 2\sin A \cos 11A]}$$

$$= \frac{\cos(7A-3A) - \cos(7A+3A) + \cos(11A-A) + \cos(11A+A)}{\sin(7A+3A) - \sin(7A-3A) + \sin(11A+A) - \sin(11A-A)}$$

$$= \frac{\cos 4A - \cos 10A + \cos 10A + \cos 12A}{\sin 10A - \sin 4A + \sin 12A - \sin 10A}$$

$$= \frac{\cos 12A + \cos 4A}{\sin 12A - \sin 4A}$$

$$= \frac{2\cos\left(\frac{12A+4A}{2}\right)\cos\left(\frac{12A-4A}{2}\right)}{2\cos\left(\frac{12A+4A}{2}\right)\sin\left(\frac{12A-4A}{2}\right)}$$

$$= \frac{\cos\left(\frac{16A}{2}\right)\cos\left(\frac{8A}{2}\right)}{\cos\left(\frac{16A}{2}\right)\sin n\left(\frac{8A}{2}\right)}$$

$$= \frac{\cos 8A \cos 4A}{\cos 8A \sin 4A}$$

$$= \frac{\cos 4A}{\sin 4A}$$

$$= \cot 4A$$

19. If $\sin x + \sin y = \frac{3}{4}$ and $\sin x - \sin y = \frac{2}{5}$. Prove that $8\tan\left(\frac{x+y}{2}\right) = 15\tan\left(\frac{x-y}{2}\right)$

Sol:

$\sin x + \sin y = \frac{3}{4}$ and $\sin x - \sin y = \frac{2}{5}$

consider, $\sin x + \sin y = \frac{3}{4}$

$$2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{3}{4} \text{ -------- } (1)$$

and

$$\sin x - \sin y = \frac{2}{5}$$

$$2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) = \frac{2}{5} \text{ -------- } (2)$$

(1) ÷ (2)

$$\frac{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)} = \frac{3/4}{2/5}$$

$$\tan\left(\frac{x+y}{2}\right)\cot\left(\frac{x-y}{2}\right) = \frac{15}{8}$$

$$8\tan\left(\frac{x+y}{2}\right)\cot\left(\frac{x-y}{2}\right) = 15 \Longrightarrow 8\tan\left(\frac{x+y}{2}\right) = \frac{15}{\cot\left(\frac{x-y}{2}\right)}$$

$$\therefore 8\tan\left(\frac{x+y}{2}\right) = 15\tan\left(\frac{x-y}{2}\right)$$

20. If $\sin x + \sin y = \frac{1}{4}$ and $\cos x + \cos y = \frac{1}{3}$. Prove that $\cot(x+y) = \frac{7}{24}$

Sol:

$\sin x + \sin y = \frac{1}{4}$ and $\cos x + \cos y = \frac{1}{3}$

consider, $\sin x + \sin y = \frac{1}{4}$

$$2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{1}{4} \text{ -------- } (1)$$

and

$$\cos x - \cos y = \frac{1}{3}$$

$$2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{2}{5} \text{ -------- } (2)$$

(1) ÷ (2)

$$\frac{2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)} = \frac{1/4}{1/3}$$

$$\frac{\sin\left(\frac{x+y}{2}\right)}{\cos\left(\frac{x+y}{2}\right)} = \frac{3}{4}$$

$$\tan\left(\frac{x+y}{2}\right) = \frac{3}{4}$$

$$\tan(x+y) = \frac{2\tan\left(\frac{x+y}{2}\right)}{1-\tan^2\left(\frac{x+y}{2}\right)} = \frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} = \frac{\frac{3}{2}}{1-\frac{9}{16}} = \frac{\frac{3}{2}}{\frac{16-9}{16}} = \frac{3}{2}\times\frac{16}{7} = \frac{24}{7}$$

$$\tan(x+y) = \frac{24}{7}$$

$$\therefore \cot(x+y) = \frac{7}{24}$$

21. If $\cos x + \cos y = \frac{3}{5}$ and $\cos x - \cos y = \frac{2}{7}$. Show that $21\tan\left(\frac{x+y}{2}\right) + 10\cot\left(\frac{x-y}{2}\right) = 0$

Sol:

$\cos x + \cos y = \frac{3}{5}$ and $\cos x - \cos y = \frac{2}{7}$

consider, $\cos x + \cos y = \frac{3}{5}$

$$2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{3}{5} \text{ -------- } (1)$$

and

$$\cos x - \cos y = \frac{2}{7}$$

$$-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) = \frac{2}{7} \text{ -------- } (2)$$

(1)÷ (2)

$$\frac{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}{-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)} = \frac{3/5}{2/7}$$

$$-\cot\left(\frac{x+y}{2}\right)\cot\left(\frac{x-y}{2}\right) = \frac{21}{10}$$

$$-10\cot\left(\frac{x+y}{2}\right)\cot\left(\frac{x-y}{2}\right) = 21 \Rightarrow -10\cot\left(\frac{x+y}{2}\right) = \frac{21}{\cot\left(\frac{x-y}{2}\right)}$$

$$-10\cot\left(\frac{x+y}{2}\right) = 21\tan\left(\frac{x-y}{2}\right)$$

$$\therefore 21\tan\left(\frac{x-y}{2}\right) + 10\cot\left(\frac{x+y}{2}\right) = 0$$

22. If $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{a+b}{a-b}$, show that $b\tan\alpha = a\tan\beta$

Sol:

Given, $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{a+b}{a-b}$

Apply Componendo and Dividendo rule

$$\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\sin(\alpha+\beta)-\sin(\alpha-\beta)} = \frac{a+b+a-b}{a+b-(a-b)}$$

$$\frac{2\sin\alpha\cos\beta}{2\cos\alpha\sin\beta} = \frac{2a}{2b}$$

$$\Rightarrow \tan\alpha\cot\beta = \frac{a}{b}$$

$$\therefore b\tan\alpha = a\tan\beta$$

23. Show that $\cos 10^0 \cos 50^0 \cos 70^0 = \frac{\sqrt{3}}{8}$

Sol:

$LHS = \cos 10^0 \cos 50^0 \cos 70^0$

$= \frac{1}{2}[2\cos 70^0 \cos 50^0] \cos 10^0$

$= \frac{1}{2}[\cos (70^0 + 50^0) + \cos (70^0 - 50^0)] \cos 10^0$

$= \frac{1}{2}[\cos 120^0 + \cos 20^0] \cos 10^0$

$= \frac{1}{4}[2\cos 120^0 \cos 10^0 + 2\cos 20^0 \cos 10^0]$

$= \frac{1}{4}[2\cos (180^0 - 60^0) \cos 10^0 + 2\cos 20^0 \cos 10^0]$

$= \frac{1}{4}[-2\cos 60^0 \cos 10^0 + \cos (20^0 + 10^0) + \cos (20^0 - 10^0)]$

$= \frac{1}{4}[-2 \times \frac{1}{2} \cos 10^0 + \cos 30^0 + \cos 10^0]$

$= \frac{1}{4} \cos 30^0$

$= \frac{1}{4} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8} = RHS$

$\therefore \cos 10^0 \cos 50^0 \cos 70^0 = \frac{\sqrt{3}}{8}$

24. If $\sin \alpha + \sin \beta = a$ and $\cos \alpha + \cos \beta = b$, then show that (i) $\tan\left(\frac{\alpha + \beta}{2}\right) = \frac{a}{b}$

(ii) $\tan (\alpha + \beta) = \frac{2ab}{b^2 - a^2}$ (iii) $\sin (\alpha + \beta) = \frac{2ab}{a^2 + b^2}$ (iv) $\cos (\alpha + \beta) = \frac{a^2 - b^2}{a^2 + b^2}$

Sol: Given $\sin \alpha + \sin \beta = a$ and $\cos \alpha + \cos \beta = b$

Consider, $\sin \alpha + \sin \beta = a$

$$2\sin\left(\frac{\alpha + \beta}{2}\right) \cos \left(\frac{\alpha - \beta}{2}\right) = a \text{ -------- } (1)$$

and $\cos \alpha + \cos \beta = b$

$$2\cos \left(\frac{\alpha + \beta}{2}\right) \cos \left(\frac{\alpha - \beta}{2}\right) = b \text{ -------- } (2)$$

(1)÷ (2)

$$\frac{2\sin \left(\frac{\alpha + \beta}{2}\right)\cos \left(\frac{\alpha - \beta}{2}\right)}{2\cos \left(\frac{\alpha + \beta}{2}\right) \cos \left(\frac{\alpha - \beta}{2}\right)} = \frac{a}{b}$$

$$\boxed{\tan \left(\frac{\alpha + \beta}{2}\right) = \frac{a}{b}}$$

(i) $\tan \left(\frac{\alpha + \beta}{2}\right) = \frac{a}{b}$

(ii) $\tan(\alpha+\beta) = \dfrac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1-\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \dfrac{2\times\frac{a}{b}}{1-\left(\frac{a}{b}\right)^2} = \dfrac{\frac{2a}{b}}{1-\frac{a^2}{b^2}} = \dfrac{\frac{2a}{b}}{\frac{b^2-a^2}{b^2}} = \dfrac{\frac{2ab^2}{b}}{b^2-a^2}$

$\therefore \tan(\alpha+\beta) = \dfrac{2\,ab}{b^2-a^2}$

(iii) $\sin(\alpha+\beta) = \dfrac{2\tan\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \dfrac{2\times\frac{a}{b}}{1+\left(\frac{a}{b}\right)^2} = \dfrac{\frac{2a}{b}}{1+\frac{a^2}{b^2}} = \dfrac{\frac{2a}{b}}{\frac{b^2+a^2}{b^2}} = \dfrac{\frac{2ab^2}{b}}{b^2+a^2}$

$\therefore \sin(\alpha+\beta) = \dfrac{2\,ab}{a^2+b^2}$

(iv) $\cos(\alpha+\beta) = \dfrac{1-\tan^2\left(\frac{\alpha+\beta}{2}\right)}{1+\tan^2\left(\frac{\alpha+\beta}{2}\right)} = \dfrac{1-\left(\frac{a}{b}\right)^2}{1+\left(\frac{a}{b}\right)^2} = \dfrac{1-\frac{a^2}{b^2}}{1+\frac{a^2}{b^2}} = \dfrac{\frac{b^2-a^2}{b^2}}{\frac{b^2+a^2}{b^2}}$

$\therefore \cos(\alpha+\beta) = \dfrac{a^2-b^2}{a^2+b^2}$