2 Polynomials

Polynomial:

An algebraic expression becomes a polynomial if the powers of variable(s) are whole numbers.

Ex: $3x - 5$, $4x^2 + x - 3$, $2x^3 + 3x^2 - 5x + 4$

Value of a polynomial

p(a) is the value of a polynomial p(x) at x = a

Ex: let $p(x) = 2x - 4$

Put $x = 1 \Longrightarrow p(1) = 2(1) - 4 = 2 - 4 = -2$

$\therefore -2$ is the value of the polynomial p(x) at x = 1

Zero of a polynomial

Zeroes of a polynomial p(x) is any real number 'k' such that p(k) = 0.

Ex: let $p(x) = x - 3$

Put $x = 3 \Longrightarrow p(3) = 3 - 3 = 0$

$\therefore$ 3 is the zero of the polynomial p(x)

Note: For finding the zeroes of the polynomial p(x), let p(x) = 0

Degree of a polynomial

The highest power of x in a polynomial p(x) is called as degree of the polynomial p(x)

Linear Polynomial

A polynomial of degree 1 is called a **linear polynomial**

Ex: $3x - 5$, $\sqrt{2}\,x + 5$, $y - \frac{4}{5}$

Quadratic Polynomial

A polynomial of degree 2 is called a **Quadratic polynomial**

Ex: $5x^2 + 4x - 1$, $x^2 + 5$, $3y^2 + 7y - \frac{4}{7}$

Cubic Polynomial

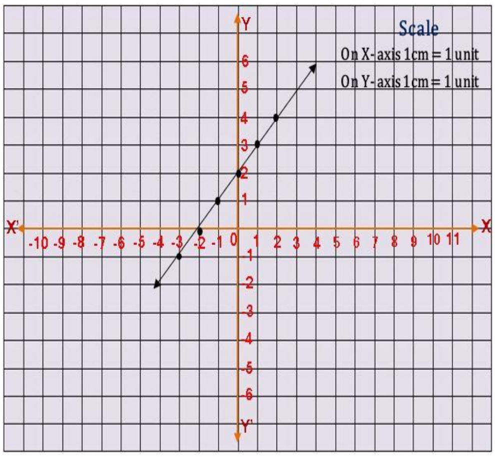
A polynomial of degree 3 is called a **Cubic polynomial**

Ex: $2x^3 + 3x^2 - 7x\ 3$, $x^3 + 5x$, $z^3 + 8z - 2$

Graphical Representation of Linear polynomial:

$y = x + 2$

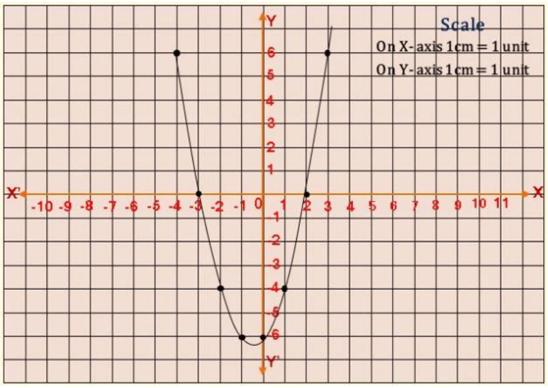
x	−3	−2	−1	0	1	2
y = x + 2	−1	0	1	2	3	4
(x, y)	(−3, −1)	(−2, 0)	(−1, 1)	(0, 2)	(1, 3)	(2, 4)



Graphical Representation of Quadratic polynomial:

$y = x^2 + x - 6$

x	−4	−3	−2	−1	0	1	2	3
$y = x^2 + x - 6$	6	0	−4	−6	−6	−4	0	6
(x, y)	(−4, 6)	(−3, 0)	(−2, −4)	(−1, −6)	(0, −6)	(1, −4)	(2, 0)	(3, 6)

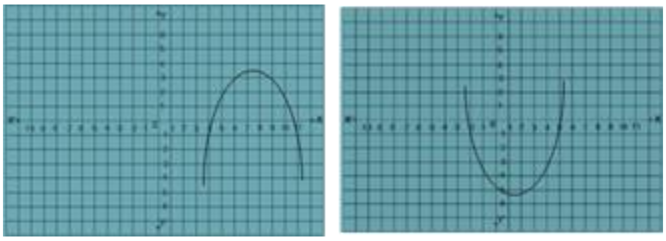


- For any quadratic polynomial $ax^2 + bx + c$, $a \neq 0$, the graph of the corresponding equation $y = ax^2 + bx + c$ $(a \neq 0,)$ either opens upwards like ∪ or opens downwards like ∩ This depends on whether $a > 0$ or $a < 0$.
  The shape of these curves is called parabolas

- The shape of the graph of $y = ax^2 + bx + c$, $(a \neq 0)$ the following three cases arise.

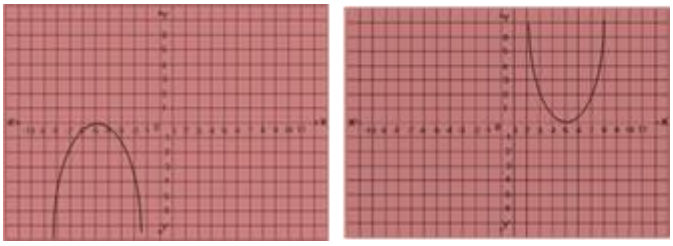
Case (i) :

Here, the graph cuts X – axis at two distinct points. In this case, the x-coordinates of those two points are the two zeroes of the quadratic polynomial $ax^2 + bx + c$. The parabola opens either upward or downward.



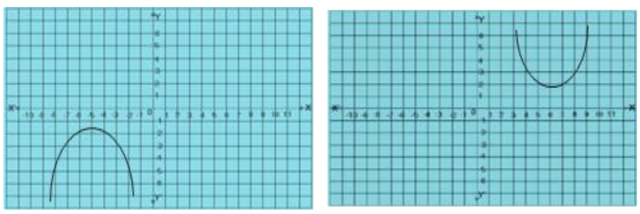
Case (ii) :

Here, the graph touches X – axis at exactly one point. In this case, the x-coordinate of that point is the only zero for the quadratic polynomial $ax^2 + bx + c$.



Case (iii) :

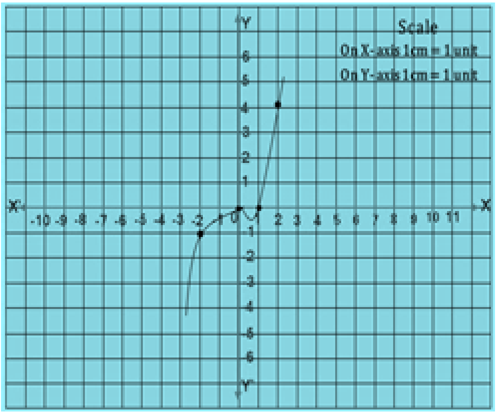
Here, the graph is either completely above the X-axis or completely below the X – axis. So, it does not cut the X-axis at any point. The quadratic polynomial $ax^2 + bx + c$ have no zero in this case.



Graphical Representation of Cubic polynomial:

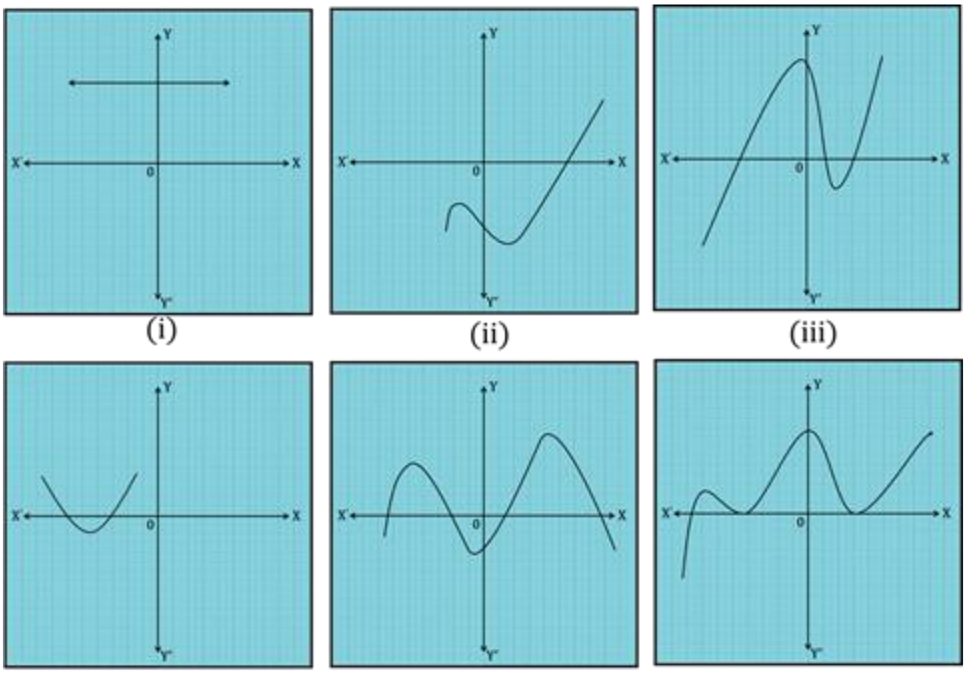
$y = x^3 - x^2$

x	−2	−1	0	1	2
$y = x^3 - x^2$	−12	−2	0	0	4
(x, y)	(−2, −12)	(−1, −2)	(0, 0)	(1, 0)	(2, 4)



EXERCISE 2.1

1. The graphs of y = p(x) are given in Fig. 2.10 below, for some polynomials p(x). Find the number of zeroes of p(x), in each case.



Solution:

(i) From the graph, the graph of polynomial is parallel to X – axis. It does not cuts X – axis at any point

∴ the polynomial has no zeroes

(ii) From the graph, the graph of polynomial is cuts the X – axis at only one point.

∴ the polynomial has one zero

(iii) From the graph, the graph of polynomial is cuts the X – axis at three points.

∴ the polynomial has three zeroes

(iv) From the graph, the graph of polynomial is cuts the X – axis at two points.
∴ the polynomial has two zeroes.

(v) From the graph, the graph of polynomial is cuts the X – axis at four points.
∴ the polynomial has four zeroes

(vi) From the graph, the graph of polynomial is cuts the X – axis at three points.
∴ the polynomial has three zeroes.

Relationship between Zeroes and Coefficients of the polynomial:

- $P(x) = ax + b$ is linear polynomial

  Zero of the polynomial is $x = -\frac{b}{a}$

- $P(x) = ax^2 + bx + c \ (a \neq 0)$ is general form of quadratic polynomial.

  Sum of the zeroes $= \alpha + \beta = -\frac{b}{a} = \frac{-(\text{coefficent of } x)}{\text{coefficent of } x^2}$

  Product of the zeroes $= \frac{c}{a} = \frac{\text{constant}}{\text{coefficent of } x^2}$

- $p(x) = ax^3 + bx^2 + cx + d \ (a \neq 0)$ is general form of cubic polynomial.

  $\alpha + \beta + \gamma = -\frac{b}{a} = \frac{-(\text{coefficent of } x^2)}{\text{coefficent of } x^3}$

  $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{\text{coefficent of } x}{\text{coefficent of } x^3}$

  $\alpha\beta\gamma = -\frac{d}{a} = \frac{-(\text{constant})}{\text{coefficent of } x^3}$

- If α, β are the zeroes of quadratic polynomial then its form is $k[x^2 - (\alpha + \beta)x + \alpha\beta]$
- If α, β and γ are the zeroes of the cubic polynomial then its form is

  $$k[x^3 - (\alpha + \beta + \gamma)x + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma]$$

EXERCISE 2.2

1. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.

   (i) $x^2 - 2x - 8$ (ii) $4s^2 - 4s + 1$ (iii) $6x^2 - 3 - 7x$ (iv) $4u^2 + 8u$
   (v) $t^2 - 15$ (vi) $3x^2 - x - 4$

Solution:

(i) Given polynomial is $x^2 - 2x - 8$

Let $P(x) = x^2 - 2x - 8$

For the zeroes of the polynomial $P(x) = 0$

$\Longrightarrow x^2 - 2x - 8 = 0$

$x^2 - 4x + 2x - 8 = 0$

$x(x - 4) + 2(x - 4) = 0$

$(x - 4)(x + 2) = 0$

$x - 4 = 0$ or $x + 2 = 0$

$x = 4$ or $x = -2$

let $\alpha = -2$ and $\beta = 4$

Sum of the zeroes $= \alpha + \beta = (-2) + 4$

$= 2$

$= \frac{-(-2)}{1}$

$= \frac{-(\text{coefficeint of } x)}{\text{coefficient of } x^2}$

Product of the zeroes $= \alpha\beta = (-2)(4)$

$= -8$

$= \frac{-8}{1}$

$= \frac{\text{constant}}{\text{coefficient of } x^2}$

(ii) Given polynomial is $4s^2 - 4s + 1$

Let $P(s) = 4s^2 - 4s + 1$

For the zeroes of the polynomial $P(s) = 0$

$\Longrightarrow 4s^2 - 4s + 1 = 0$

$s^2 - 2s - 2s + 1 = 0$

$s(2s - 1) - 1(2s - 1) = 0$

$(2s - 1)(2s - 1) = 0$

$2s - 1 = 0$ or $2s - 1 = 0$

$s = \frac{1}{2}$ or $s = \frac{1}{2}$

let $\alpha = \frac{1}{2}2$ and $\beta = \frac{1}{2}$

Sum of the zeroes $= \alpha + \beta = \frac{1}{2} + \frac{1}{2}$

$= 1$

$= \frac{-(-4)}{4}$

$= \frac{-(\text{coefficeint of } s)}{\text{coefficient of } s^2}$

Product of the zeroes $= \alpha\beta = \frac{1}{2} \times \frac{1}{2}$

$= \frac{1}{4}$

$= \frac{\text{constant}}{\text{coefficient of } s^2}$

(iii) Given polynomial is $6x^2 - 3 - 7x$

Let $P(x) = 6x^2 - 7x - 3$

For the zeroes of the polynomial $P(x) = 0$

$\Longrightarrow 6x^2 - 7x - 3 = 0$

$6x^2 - 9x + 2x - 3 = 0$

$3x(2x - 3) + 1(2x - 3) = 0$

$(2x - 3)(3x + 1) = 0$

$2x - 3 = 0$ or $3x + 1 = 0$

$x = \frac{3}{2}$ or $x = \frac{-1}{3}$

let $\alpha = \frac{-1}{3}$ and $\beta = \frac{3}{2}$

Sum of the zeroes $= \alpha + \beta = \frac{-1}{3} + \frac{3}{2}$

$= \frac{-2+9}{6} = \frac{7}{6}$

$= \frac{-(-7)}{6}$

$= \frac{-(\text{coefficeint of x})}{\text{coefficient of } x^2}$

Product of the zeroes $= \alpha\,\beta = \frac{-1}{3} \times \frac{3}{2}$

$= \frac{-3}{6}$

$= \frac{-1}{2}$

$= \frac{\text{constant}}{\text{coefficient of } x^2}$

(iv) Given polynomial is $4u^2 + 8u$

Let $P(u) = 4u^2 + 8u$

For the zeroes of the polynomial $P(u) = 0$

$\Longrightarrow 4u^2 + 8u = 0$

$4u(u+2) = 0$

$4u = 0$ or $u + 2 = 0$

$u = 0$ or $u = -2$

let $\alpha = -2$ and $\beta = 0$

Sum of the zeroes $= \alpha + \beta = (-2) + 0$

$= -2$

$= \frac{-(8)}{4}$

$= \frac{-(\text{coefficeint of u})}{\text{coefficient of } u^2}$

Product of the zeroes $= \alpha\,\beta = (-2)(0)$

$= 0$

$= \frac{0}{4}$

$= \frac{\text{constant}}{\text{coefficient of } u^2}$

(v) Given polynomial is $t^2 - 15$

Let $P(t) = t^2 - 15$

For the zeroes of the polynomial $P(t) = 0$

$\Longrightarrow t^2 - 15 = 0$

$t^2 = 15$

$t = \pm\sqrt{15}$

$t = +\sqrt{15}$ or $t = -\sqrt{15}$

let $\alpha = -\sqrt{15}$ and $\beta = \sqrt{15}$

Sum of the zeroes $= \alpha + \beta = (-\sqrt{15}) + \sqrt{15}$

$= 0$

$= \frac{-(0)}{1}$

$= \frac{-(\text{coefficeint of t})}{\text{coefficient of } t^2}$

Product of the zeroes $= \alpha\,\beta = (-\sqrt{15})(\sqrt{15})$

$= -\sqrt{15}$

$= \frac{-\sqrt{15}}{1}$

$= \frac{\text{constant}}{\text{coefficient of } t^2}$

(vi) Given polynomial is $3x^2 - x - 4$

Let $P(x) = 3x^2 - x - 4$

For the zeroes of the polynomial $P(x) = 0$

$\Rightarrow 3x^2 - x - 4 = 0$

$3x^2 - 4x + 3x - 4 = 0$

$x(3x - 4) + 1(3x - 4) = 0$

$(3x - 4)(x + 1) = 0$

$3x - 4 = 0$ or $x + 1 = 0$

$x = \frac{4}{3}$ or $x = -1$

let $\alpha = -1$ and $\beta = \frac{4}{3}$

Sum of the zeroes $= \alpha + \beta = -1 + \frac{4}{3}$

$= \frac{-3 + 4}{3}$

$= \frac{-1}{3}$

$= \frac{-(1)}{3}$

$= \frac{-(\text{coefficeint of } x)}{\text{coefficient of } x^2}$

Product of the zeroes $= \alpha\beta = (-1)(\frac{4}{3})$

$= \frac{-4}{3}$

$= \frac{\text{constant}}{\text{coefficient of } x^2}$

2. Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.

(i) $\frac{1}{4}, -1$ (ii) $\sqrt{2}, \frac{1}{3}$ (iii) $0, \sqrt{5}$ (iv) $1, 1$ (v) $-\frac{1}{4}, \frac{1}{4}$ (vi) $4, 1$

Solution:

(i) Given $\alpha + \beta = \frac{1}{4}$ and $\alpha\beta = -1$

Quadratic polynomial is $k[x^2 - (\alpha + \beta)x + \alpha\beta]$

$= k[x^2 - (\frac{1}{4})x + (-1)]$

$= k[x^2 - \frac{x}{4} - 1]$

When $k = 4$, the quadratic polynomial is $4x^2 - x - 4$

(ii) Given $\alpha + \beta = \sqrt{2}$ and $\alpha\beta = \frac{1}{3}$

Quadratic polynomial is $k[x^2 - (\alpha + \beta)x + \alpha\beta]$

$= k[x^2 - (\sqrt{2})x + \frac{1}{3}]$

$= k[x^2 - \sqrt{2}x + \frac{1}{3}]$

When $k = 3$, the quadratic polynomial is $3x^2 - 3\sqrt{2}x + 1$

(iii) Given $\alpha + \beta = 0$ and $\alpha\beta = \sqrt{5}$

Quadratic polynomial is $k[x^2 - (\alpha + \beta)x + \alpha\beta]$

$= k[x^2 - (0)x + \sqrt{5}]$

$= k[x^2 + \sqrt{5}]$

When $k = 1$, the quadratic polynomial is $x^2 + \sqrt{5}$

(iv) Given $\alpha + \beta = 1$ and $\alpha\beta = 1$

Quadratic polynomial is $k[x^2 - (\alpha + \beta)x + \alpha\beta]$

$= k[x^2 - (1)x + 1]$

$= k[x^2 - x + 1]$

When $k = 1$, the quadratic polynomial is $x^2 - x + 1$

(v) Given $\alpha + \beta = -\frac{1}{4}$ and $\alpha\beta = \frac{1}{4}$

Quadratic polynomial is $k[x^2 - (\alpha + \beta)x + \alpha\beta]$

$= k[x^2 - (-\frac{1}{4})x + \frac{1}{4}]$

$= k[x^2 + \frac{x}{4} + \frac{1}{4}]$

When $k = 4$, the quadratic polynomial is $4x^2 + x + 1$

(vi) Given $\alpha + \beta = 4$ and $\alpha\beta = 1$

Quadratic polynomial is $k[x^2 - (\alpha + \beta)x + \alpha\beta]$

$= k[x^2 - (4)x + 1]$

$= k[x^2 - 4x + 1]$

When $k = 1$, the quadratic polynomial is $x^2 - 4x + 1$